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Variational principles for nonlinear fiber optics

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Abstract

A family of variational principles with a free parameter is obtained for nonlinear fiber optics by the semi-inverse method proposed by Ji-Huan He [He JH. Variational principle for some nonlinear partial differential equations with variable coefficients. Chaos, Solitons & Fractals 2004;19:847].

1. Introduction

In recent years, the observation of temporal solitons in optical fiber has resulted in a great amount of research, and which has been motivated by both fundamental interest and the potential for applications in telecommunications. In theoretic research, variational principles have come to play an important role in nonlinear fiber optics and related fields, and which have served as a basis for development of variety of approximate methods of analysis and various numerical techniques [1]. The application of variational theory to nonlinear optics was initiated by Anderson [2], after that, there are a huge number of papers using variational approximations to nonlinear optics. Ji-Huan He [3] obtained some variational principles for some nonlinear partial differential equations with variable coefficients by the semi-inverse method [4], which reveals to be much more effective than the Noether's theorem as illustrated in Ref. [5]. In this paper, we want to obtain a nonlinear partial differential equation with a free parameter in nonlinear fiber optics by the semi-inverse method [3,4].

2. Variational model

Though there exists a huge body of literature on variational approximations, the establishment of variational formulae for nonlinear optics is not dealt with so much.

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The typical and most important equation for nonlinear optics is the nonlinear Schrödinger equation (NLS) [1,6], which governs the propagation of an electromagnetic wave in a glass fiber, or the spatial evolution of the electromagnetic field in a planar waveguide.

The NLS equation for slowly varying amplitude $\Psi(z,\tau)$ reads

$$\mathrm{i}\Psi_z + \frac{1}{2}D\Psi_{\tau\tau} + \gamma|\Psi|^2\Psi = 0, \tag{1}$$

where $\tau = t - z/V_{\mathrm{gr}}$ is the "reduced time"; z and t are the propagation distance along the fiber and time respectively; V_{gr} is the carrier's group velocity; D is the dispersion coefficient.

On substituting $\Psi = \mu + \mathrm{i}\upsilon$, where μ and υ are the envelope components along the principal axes, and which are the real functions of z and τ. The real part of Eq. (1) can be written as the following differential equation for μ and υ:

$$-\upsilon_z + \frac{1}{2}D\mu_{\tau\tau} + \gamma(\mu^2 + \upsilon^2)\mu = 0, \tag{2}$$

In the same way, the imaginary part of Eq. (1) is

$$\mu_z + \frac{1}{2}D\upsilon_{\tau\tau} + \gamma(\mu^2 + \upsilon^2)\upsilon = 0. \tag{3}$$

In order to establish an variational formulation, by the semi-inverse method [3,4], we construct the following trial-functional

$$J(\mu,\upsilon) = \int \left[\lambda\upsilon\mu_z - (1-\lambda)\upsilon_z\mu - \frac{1}{4}D\mu_\tau^2 + \frac{1}{2}\gamma(\mu^4 + 2\mu^2\upsilon^2) + F\right]\mathrm{d}z\,\mathrm{d}\tau, \tag{4}$$

where λ is an arbitrary parameter, F is an unknown function of υ and its derivatives.

There are many alternative approaches to construction of trial-functional, illustrating examples can be found in Refs. [7–14]. The advantage of the above trial-functional is that the stationary condition with respect to μ results in Eq. (2).

Calculating the functional, Eq. (4), with respect to υ, we obtain the following Euler–Lagrange equation:

$$\mu_z + \gamma\mu^2\upsilon + \frac{\delta F}{\delta\upsilon} = 0, \tag{5}$$

where $\delta F/\delta\upsilon$ is called variational derivative with respect to υ, defined as

$$\frac{\delta F}{\delta\upsilon} = \frac{\partial F}{\partial\upsilon} - \frac{\partial}{\partial z}\frac{\partial F}{\partial\mu_z} + \frac{\partial^2}{\partial\tau^2}\frac{\partial F}{\partial\mu_{\tau\tau}} - \cdots. \tag{6}$$

We search for such an F so that Eq. (5) becomes Eq. (3). Accordingly we set

$$\frac{\delta F}{\delta\upsilon} = -\mu_z - \gamma\mu^2\upsilon = \frac{1}{2}D\upsilon_{\tau\tau} + \gamma\upsilon^3, \tag{7}$$

from which the unknown F can be determined as follows

$$F = -\frac{1}{4}D\upsilon_\tau^2 + \frac{1}{4}\gamma\upsilon^4. \tag{8}$$

We, therefore, obtain the following needed variational principle:

$$J(\mu,\upsilon) = \int L_\lambda\,\mathrm{d}z\,\mathrm{d}\tau, \tag{9a}$$

where the Lagrange multiplier is defined as

$$L_\lambda = \lambda\upsilon\mu_z - (1-\lambda)\upsilon_z\mu - \frac{1}{4}D(\mu_\tau^2 + \upsilon_\tau^2) + \frac{1}{4}\gamma(\mu^2 + \upsilon^2)^2. \tag{9b}$$

It is easy to prove that stationary conditions of the above functional are Eqs. (2) and (3).

According to the theorem in [3,14], if $L_\lambda(\mu,\upsilon)$ is a known Lagrangian as defined above, the Euler–Lagrange equations keep unchanged if we add a term, $a\mu\mu_z + b\upsilon\upsilon_z$, to the Lagrangian. Therefore, the Lagrangian, Eq. (9b), can be written equivalently in the form

$$L_\lambda = \lambda\upsilon\mu_z - (1-\lambda)\upsilon_z\mu - \frac{1}{4}D(\mu_\tau^2 + \upsilon_\tau^2) + \frac{1}{4}\gamma(\mu^2 + \upsilon^2)^2 + a\mu\mu_z + b\upsilon\upsilon_z, \tag{10}$$

where a and b are arbitrary constants.

By choice of $a = -\mathrm{i}\lambda$ and $b = \mathrm{i}(\lambda - 1)$, the Lagrangian, Eq. (10), can be written in the form of Ψ

$$L_\lambda(\Psi) = \mathrm{i}\left(\frac{1}{2} - \lambda\right)\Psi\Psi_z - \mathrm{i}\frac{1}{2}\Psi\Psi_z^* - \frac{1}{4}D|\Psi_z|^2 + \frac{1}{4}\gamma|\Psi|^4, \tag{11}$$

where $\Psi_z^* = \mu_z - i\upsilon_z$. Choosing $\lambda = 0, 1/2$, and 1, we can obtain the following useful functions respectively:

$$L_0(\Psi) = \mathrm{i}\frac{1}{2}(\Psi\Psi_z - \Psi\Psi_z^*) - \frac{1}{4}D|\Psi_z|^2 + \frac{1}{4}\gamma|\Psi|^4, \tag{12}$$

$$L_{1/2}(\Psi) = -\mathrm{i}\frac{1}{2}\Psi\Psi_z^* - \frac{1}{4}D|\Psi_z|^2 + \frac{1}{4}\gamma|\Psi|^4, \tag{13}$$

$$L_1(\Psi) = -\mathrm{i}\frac{1}{2}(\Psi\Psi_z + \Psi\Psi_z^*) - \frac{1}{4}D|\Psi_z|^2 + \frac{1}{4}\gamma|\Psi|^4. \tag{14}$$

In case $\lambda = 1/2$, we can also obtain the following Lagrangian:

$$\tilde{L}_{1/2}(\Psi) = \mathrm{i}c\Psi\Psi_z - \mathrm{i}d\Psi\Psi_z^* - \frac{1}{4}D|\Psi_z|^2 + \frac{1}{4}\gamma|\Psi|^4, \tag{15}$$

where c and d are arbitrary constants, satisfying the identity $c + d = 1/2$. In case $c = d = 1/4$, we obtain

$$\tilde{\tilde{L}}_{1/2}(\Psi) = \mathrm{i}\frac{1}{4}(\Psi\Psi_z - \Psi\Psi_z^*) - \frac{1}{4}D|\Psi_z|^2 + \frac{1}{4}\gamma|\Psi|^4. \tag{16}$$

Eq. (16) agrees with Malomed's result (see Eq. (15) in Ref. [1]). Therefore Eq. (11) has a universality in some sense for nonlinear fiber optics.

3. Conclusion

Ji-Huan He's semi-inverse method, which is proved to be a powerful mathematical tool to the search for variational formulations directly from the field equations, can be easily extended to any nonlinear fiber optics, and the present letter can be used as paradigms for many other applications in searching for variational principles for real-life physics problems.

References

[1] Malomed BA. Variational methods in nonlinear fiber optics and related fields. In: Wolf E, editor. Progress in optics, vol. 43. Elsevier Science B.V; 2002. [Chapter 2].
[2] Anderson D. Phys Rev A 1983;27:31–5.
[3] He JH. Variational principle for some nonlinear partial differential equations with variable coefficients. Chaos, Solitons & Fractals 2004;19:847–51.
[4] He JH. Semi-inverse method of establishing generalized variational principles for fluid mechanics with emphasis on turbomachinery aerodynamics. Int J Turbo Jet Engines 1997;14(1):23–8.
[5] Khater AH, Moussa MHM, Abdul-Aziz SF. Invariant variational principles and conservation laws for some nonlinear partial differential equations with constant coefficients-II. Chaos, Solitons & Fractals 2003;15:1–13.
[6] Li B, Chen Y. On exact solution of the nonlinear Schrödinger equations in optical fiber. Chaos, Solitons & Fractals 2004;21:241–7.
[7] He JH. A classical variational model for micropolar elastodynamics. Int J Nonlinear Sci Numer Simul 2000;1(2):133–8.
[8] He JH. Variational theory for linear magneto-electro-elasticity. Int J Nonlinear Sci Numer Simul 2001;2(4):309–16.
[9] Hao TH. Application of the Lagrange multiplier method by the semi-inverse method to the search for generalized variational principle in quantum mechanics. Int J Nonlinear Sci Numer Simul 2003;4(3):311–2.
[10] Liu HM. Variational approach to nonlinear electrochemical system. Int J Nonlinear Sci Numer Simul 2004;5(1):95–6.
[11] He JH. Hamilton principle and generalized variational principles of linear thermopiezoelectricity. ASME J Appl Mech 2001;68(4):666–7.
[12] He JH. Generalized Hellinger–Reissner principle. ASME J Appl Mech 2000;67(2):326–31.
[13] He JH. Coupled variational principles of piezoelectricity. Int J Eng Sci 2000;39(3):323–41.
[14] He JH. Generalized variational principles in fluids. Hong Kong: Science & Culture Publishing House of China; 2003. [in Chinese].