A Detailed Examination of the Translation Efficiency of Fiber Strength into Composite Strength

NING PAN
Division of Textiles and Clothing
Biological and Agricultural Engineering Department
University of California
Davis, CA 95616

ABSTRACT: This work deals with the relationships between the strengths of fiber, fiber bundle and a unidirectionally reinforced fiber composite. A factor designated as the surviving fiber ratio is introduced in this study to reflect the fact that during the fracture process of a fibrous structure, the fibers will not break simultaneously because of the fiber strength variation. This new factor results in a gradual breakage pattern on the stress-strain curve, and hence reduces both the ultimate strength and the breaking strain of the structure. Incorporating this new factor into analysis leads to a more realistic prediction.

Using the previous results on the distributions of fiber and fiber bundle strengths, the distribution function of the composite strength and the related distribution parameters are then derived. The effects of the interactions between fibers and matrix in the composite reflected by the critical fiber length, and the fiber strength variations accounted for by the surviving fiber ratio are included when calculating the distribution parameters for the composite strength.

Next, the comparison between the predicted stress-strain curves of fiber, fiber bundle, and composite is provided to reveal the important mechanisms influencing composite strength. The most probable strength is then derived as the best estimate of the actual strength for these fibrous systems. This actual composite strength σ_c is compared with the mean composite strength $\bar{\sigma}_c$, the actual fiber bundle strength σ_p, and the actual fiber strength σ_f so that the translation efficiency of fiber strength into composite strength is described.

Based on the new approach, the important issues such as the fragmentation and the experimentally observed synergetic effects on composite strength are analyzed in detail, and the necessary conditions for these effects to occur are provided. The influences on the composite strength of the fiber and matrix properties, including the fiber scale and shape parameters α, β, the tensile moduli ratio E_m/E_f, the fiber volume fraction V_f and the shear yielding stress τ_y of the interface, are also discussed.

KEY WORDS: distribution of composite strength, fiber strength variation, stress-strain curve, mode of the strength, fiber-matrix interactions, synergetic effect.

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1. INTRODUCTION

THE STRENGTH AND toughness of a fiber composite are influenced to a high degree by the properties of the composite interface through which interaction between fibers and the matrix is realized during composite fracture process. The most significant evidence of this fiber-matrix interaction is best revealed by the existence of the so-called fragmentation phenomenon in the composite, described first by Kelly and Tyson [1].

A unidirectional composite is made of parallel continuous fiber bundles impregnated into matrix material. If there were no interaction between fibers and the matrix, prediction of the composite strength would be a simple matter of using the Rule of Mixtures, provided that the difference between the breaking elongation of the two constituents is taken into account. In other words, the composite strength would be linearly proportional to that of the fiber bundle. The occurrence of the fiber-matrix interaction, however, greatly complicates this otherwise very direct relationship.

There have been a great deal of theoretical and experimental studies on the relationship between fiber strength and composite strength, and reference to some of the representative papers [2–11] is provided in this article.

One of these studies is particularly interesting to the present analysis. It is a relatively thorough experimental work done by Bader and Priest [2] involving tensile strength examination of single fibers, fiber bundles, impregnated bundles and hybrid bundles. Their results were later utilized and further analyzed by Watson and Smith [10]. For convenience, we provide in Table 1 from that work a portion of the data relevant to the present study.

In that experimental work, the same single fibers were arranged in parallel to form the fiber bundles, and the fiber bundles were then embedded into matrix material to form the unidirectional composites. The strengths of the single fibers, the fiber bundles and the composites were then tested as shown in the table.

The following general observations can be made based on the experimental results:

1. As being well-known, for the same gauge length, the mean strength of the single fiber is greater than that of the fiber bundle.
2. For the same gauge length, the mean strength of the composite is greater than that of the fiber bundle (in the example, even greater than that of the fiber). This is obviously in contradiction to the Rule of Mixtures which states that a composite strength cannot exceed that of its strongest constituent. The data

Table 1. The strengths tested at gauge length 20 mm [10].

Type	Single Fibers	Fiber Bundles	Composites
Test number	70	25	28
Mean strength (GPa)	2.45	1.68	2.82
Strength S.D. (GPa)	0.49	0.10	0.16

therefore provide a clear evidence that the interaction between matrix and fibers does have a synergetic function, leading to a system with strength higher than its constituents. Considering the fact that the composite strength is a result of the fiber strength discounted by the fiber volume fraction, this synergetic result is even more significant.

3. For the same gauge length, the ranking in terms of the strength standard deviation is single fiber > composite > fiber bundle.

The present study is an attempt to examine and explain in detail the above strength discrepancies using statistical analysis with the inclusion of composite interfacial property, the fiber property variations, and the interaction between fiber and matrix. The fiber-matrix interaction may take place either during composite manufacturing process, where the liquid epoxy may alter the fiber property by penetrating into the pores on the fiber surface, and the relatively high temperature may also promote property changes of fibers and matrix, or during the composite extension upon loading, where the interaction is mainly through stress transfer between fiber and matrix via the interface as characterized by the fragmentation process. The present analysis will however ignore the effects caused by the property changes of the fiber and matrix during composite manufacturing process.

The following assumptions are made in the present work:

1. Fiber strength distribution is of Weibull form.
2. Fibers have linear stress-strain relationship up to breakage.
3. The composite is a unidirectional lamina with continuous fibers arranged parallel to the loading direction. The effect of fiber misorientation has been studied quite thoroughly and is therefore excluded here.
4. Variations, and changes during composite extension, of the interfacial properties between fiber and the matrix are negligible.
5. The interactions between fibers and matrix in a composite will not affect the form of the strength distribution of the individual fibers.
6. When a fiber breaks, the load it was carrying is equally shared among the surviving fibers. The effects of stress concentration and dynamic wave propagation are ignored.

2. THE STATISTICAL DISTRIBUTION OF FIBER BUNDLE STRENGTH

The stress-strain curve of a fiber bundle would be identical to that of its constituent fibers if all fibers were uniform in their tensile properties. Unfortunately, in reality, there is more or less a dispersion in the fiber mechanical behavior, this fiber property dispersion will inevitably lead to a discrepancy between the properties of the single fiber and the fiber bundle.

2.1. The Statistical Distribution of Fiber Strength

According to Coleman [12], the cumulative strength probability distribution function of a fiber is of Weibull type. So for a fiber with length l_f, the probability of the fiber strength being σ_f is

$$F(\sigma_f) = 1 - \exp[-l_f \alpha \sigma_f^\beta] \tag{1}$$

where α is the scale parameter and β is the shape parameter of the fiber and both are independent of the fiber length l_f. The shape parameter β is an indicator of the fiber strength variation. A higher β value corresponds a lower variation, and when $\beta \rightarrow \infty$, the fiber variation would approach zero and the fiber strength would be independent of its length.

The mean or the expected value of the fiber strength $\bar{\sigma}_f$ can then be calculated as

$$\bar{\sigma}_f = (l_f \alpha)^{-1/\beta} \Gamma\left(1 + \frac{1}{\beta}\right) \tag{2}$$

where $\Gamma()$ is the Gamma function, and the standard deviation of the strength is

$$\Theta_f = \bar{\sigma}_f \left(\frac{\Gamma\left(1 + \frac{2}{\beta}\right)}{\Gamma^2\left(1 + \frac{1}{\beta}\right)} - 1 \right)^{1/2} \tag{3}$$

Note that, as pointed out in Reference [10], Equation (1) may not always be accurate for some fibers; it may overestimate the strength for shorter fiber length while underestimating the strength for longer length. However, we have assumed that the present analysis focuses only on the fibers whose strength distributions are strictly Weibull forms.

2.2. The Statistical Distribution of Fiber Bundle Strength

Let us then consider a fibrous system where N fibers form a parallel bundle with no interaction between individual fibers. Because of the variation in single fiber strength, the fiber bundle strength will consequently obey a statistical distribution as well. This problem was first tackled by Daniels in Reference [13]. Based on his analysis, for a large bundle of high N value, the density distribution function of the bundle strength σ_p approaches a normal form

$$H(\sigma_p) = \frac{1}{\sqrt{2\pi}\Theta_p} \exp\left[- \frac{(\sigma_p - \bar{\sigma}_p)^2}{2\Theta_p^2} \right] \tag{4}$$

where $\bar{\sigma}_p$ is the expected value of the bundle strength

$$\bar{\sigma}_p = (l_f \alpha \beta)^{-1/\beta} \exp\left(-\frac{1}{\beta}\right) \tag{5}$$

and Θ_p is the standard deviation of the strength

$$\Theta_p^2 = (l_f\alpha\beta)^{-2/\beta}\left[\exp\left(-\frac{1}{\beta}\right)\right]\left[1 - \exp\left(-\frac{1}{\beta}\right)\right]\cdot N^{-1} \tag{6}$$

It is already well recognized that the expected strength of a fiber bundle is lower than that of the fiber. This can be seen by comparing Equations (2) and (5). The strength variation of the fiber bundle is also smaller, depending among other factors on its size N, than that of the fiber given in Equation (3).

It is apparent that the tensile modulus of this fiber bundle is identical to that of the fiber.

3. THE STRENGTH DISTRIBUTION OF A FIBER REINFORCED COMPOSITE

It has become a common knowledge that fibers, once embedded into a matrix material to form a composite, will behave differently due to the interaction between them and the matrix, and this interaction will inevitably alter the properties of the fibers.

3.1. The Fragmentation Process, the Critical Length and the *in situ* Strength of Fibers

It has been frequently observed during the fracture process of composites as well as of blended yarns [16] that, so long as there is a difference between the breaking strains of the constituents, the fibers will break repeatedly with increasing strain of the structure until the length of the fiber fragments reaches a minimum value where load can no longer build up to its broken strength. This length is well known as the critical length. If σ_f^b is the tensile stress which causes the fiber to break, it follows that this critical length l_c is given by Reference [17] as

$$l_c = \frac{r_f\sigma_f^b}{\tau_y} \tag{7}$$

where r_f is the fiber radius and τ_y is the yielding shear strength of the matrix adjacent to the interface or that of the fiber-matrix interface, whichever is less. It is implied in this equation, as pointed out by Rosen [15], that the original fiber, once embedded in the matrix, has to be treated as a chain of statistically independent fiber segments of length l_c whose value is determined by the fiber *in situ* strength, fiber size and the quality of the interface between the fiber and the matrix as specified in the equation.

Furthermore, as indicated by Henstenburg, Netravali and Phoenix [18,19], the actual fragments lengths are not a constant and vary in the range of $l_c/2$ to l_c. This problem can be solved by replacing l_c with the mean fragment length $3l_c/4$ if a uniform distribution of this length is assumed. For simplicity however, we still use Equation (7) for this study.

Additionally, because of the strong strength-length dependence of fibers, the fiber strength σ_f^* can no longer be treated as a constant. The fragmentation process has revealed that during the extension of the composite, the fibers are stretched segment by segment through matrix. So according to the Weibull rule, the strengths of these fiber segments will become higher owing to their shorter lengths.

During the composite extension, by definition any fiber fragment with length longer than l_c is still able to break somewhere along its center section as its stress exceeds its current strength of σ_f^*. Therefore, the mean length before fibers break into l_c will be $4l_c/3$. This will be the length by which the value of σ_f^* for the new fiber fragment is determined. Keeping this in mind and combining Equations (2) and (7) gives

$$l_c = \left[\frac{r_f\left(\frac{4}{3}\alpha\right)^{-1/\beta}\Gamma\left(1+\frac{1}{\beta}\right)}{\tau_y}\right]^{\beta/(1+\beta)} \tag{8}$$

The *in situ* fiber strength σ_f^* can then be determined from Equation (2) replacing the original fiber length l_f by the critical fiber length l_c to reflect the fiber-matrix interactions.

3.2. The Strength Distribution of a Composite

In view of the preceding analysis, the strength σ_c of the composite under uniaxial loading has also to be treated as a statistical variable. Assuming the fibers to have lower breaking elongation, the composite strength can be readily obtained from

$$\sigma_c = V_f\sigma_p + V_mE_m\epsilon_p \tag{9}$$

where E_m is the matrix modulus, V_f and V_m are the volume fractions of the fiber and matrix respectively, and ϵ_p is the breaking strain of the fiber bundle and is related to the bundle strength σ_p through the tensile modulus of the bundle which is identical to that of the fiber. So the above equation can be reduced into

$$\sigma_c = V_f\sigma_p + \frac{V_mE_m}{E_f}\sigma_p = \left(V_f + \frac{V_mE_m}{E_f}\right)\sigma_p \tag{10}$$

Equation 10 states that the statistical variable σ_c is proportional to the statistical variable σ_p whose distribution is defined in Equation (4). Because of the normality of σ_p, σ_c is hence a normal variable as well with the parameters, according to statistical theory, as

$$\bar{\sigma}_c = \left(V_f + \frac{V_mE_m}{E_f}\right)\bar{\sigma}_p(l_c) \tag{11}$$

and

$$\Theta_c = \left(V_f + \frac{V_m E_m}{E_f}\right)\Theta_p(l_c) \tag{12}$$

where $\bar{\sigma}_p(l_c)$ and $\Theta_p(l_c)$ are the mean fiber bundle strength and its standard deviation which can be calculated from Equations (5) and (6), except that the fiber length l_f in the equations has to be replaced by the critical length l_c defined in Equation (8). This step is crucial so that the effect of fiber-matrix interaction can be included.

Then the distribution density function of the composite strength can be expressed as

$$H(\sigma_c) = \frac{1}{\sqrt{2\pi}\Theta_c}\exp\left[-\frac{(\sigma_c - \bar{\sigma}_c)^2}{2\Theta_c^2}\right] \tag{13}$$

If we accept the hypothesis on estimating the maximum range of a statistical variable, based on the normality of the composite strength distribution, there is over 99% chance that the actual composite strength will fall into the range of $\bar{\sigma}_c \pm 3\Theta_c$, i.e.,

$$\sigma_c \sim \bar{\sigma}_c \pm 3\Theta_c = \left(V_f + \frac{V_m E_m}{E_f}\right)[\bar{\sigma}_p(l_c) \pm 3\Theta_p(l_c)] \tag{14}$$

4. THE STRESS-STRAIN CURVES OF FIBER BUNDLES AND COMPOSITES

Suppose the single fibers used for this study are of linear mechanical behavior prior to failure. The fiber properties such as the tensile modulus, the mean strength and breaking strain are provided in Table 2.

Table 2. Properties used for calculation.

Item	Fiber	Matrix	Unit
Fiber radius r_f	5×10^{-3}		mm
Fiber gauge length l_f	10		mm
Tensile modulus	$E_f = 72.0$	$E_m = 1.44$	GPa
Fiber number N	200		
Fiber shape parameter β	4.0		
Fiber scale parameter α	2×10^{-3}		1/(mm•GPa$^\beta$)
Fiber mean breaking strain $\bar{\epsilon}_f$	3.35×10^{-2}		
Mean fiber strength $\bar{\sigma}_f$	2.41		GPa
Fiber volume fraction V_f	0.6		
Interfacial shear yielding stress τ_y	0.8		GPa

4.1. The Stress-Strain Relationship of Fiber Bundle

For a fiber bundle of length l_f under a given external strain ϵ_p high enough to cause fiber breakage, the fibers will not fail at the same time because of the variations between fiber strengths. Instead, they will break gradually, according to their strength distribution, over a range of the strain ϵ_p. Therefore, the fiber bundle stress $\sigma_p(l_f)$ can be expressed as

$$\sigma_p(l_f) = \Psi_f(l_f)E_f\epsilon_p \tag{15}$$

where $\Psi_f(l_f)$ is the fraction of the number of fibers that are not broken yet at the current strain level ϵ_p and are still carrying the load. This surviving fiber ratio can be calculated as

$$\Psi_f(l_f) = \int_{E_f\epsilon_p}^{\infty} H(\sigma_p)d\sigma_p \tag{16}$$

where $H(\sigma_p)$ is the distribution density function of the fiber bundle defined in Equation (4).

4.2. The Stress-Strain Relationship of the Composite

Likewise, the conventional stress-strain relationship (the model I) of a fiber composite

$$\sigma_c(l_f) = [E_fV_f + (1 - V_f)E_m]\epsilon_c \tag{17}$$

has to be modified into (the model II)

$$\sigma_c(l_c) = [\Psi_f(l_c)E_fV_f + (1 - V_f)E_m]\epsilon_c \tag{18}$$

The surviving fiber ratio $\Psi_f(l_c)$ is identical to that defined in Equation (16), but the fiber length l_f has to be replaced by the critical length l_c when constructing the distribution function $H(\sigma_p)$ so as to reflect the fiber-matrix interactions in the composite.

The comparison between the stress-strain curves of the single fiber, the fiber bundle, and the fiber reinforced composite according to Equations (17) and (18), using the data in Table 2, is provided in Figure 1. It is seen from the figure that, compared with others, the fiber bundle possesses both the lowest strength and breaking elongation due to the fiber strength variations represented by the surviving fiber ratio $\Psi_f(l_f)$. Yet once the fiber bundle is embedded into a matrix to form a composite, Equations (17) and (18) give two significantly different results. According to Equation (17), the model I where the effects associated with both the fiber strength variations (the factor Ψ_f) and the fiber-matrix interaction (the critical length l_c) are excluded, the composite failure takes place at the same breaking elongation as that of the fiber, higher than that of the fiber bundle; therefore the

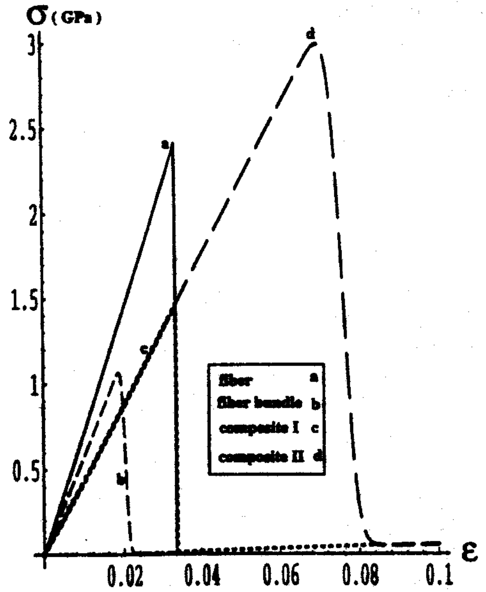


Figure 1. The stress-strain curves of fiber, fiber bundle, and composite based on two theories.

strength of the composite is also higher than that of the fiber bundle. But using Equation (18), the model II, the breaking elongation and the strength of the composite are seen to be the greatest among all the cases. This is obviously due to the reinforcing mechanism of the fiber-matrix interaction reflected by the critical fiber length l_c which is so dominant that it compensates for the effect of the fiber strength variations $\Psi_f(l_c)$, which as shown above reduces both the strength and breaking elongation of a structure. Further detailed comparisons and explanations between these cases will be given in the later sections.

Finally, the effect of the surviving fiber ratio Ψ_f blurs the otherwise sharp edges of the stress-strain curves of the fiber bundle and the composite (model II), yielding results closer to the real situations.

5. THE ACTUAL STRENGTHS AND BREAKING ELONGATIONS OF FIBERS, FIBER BUNDLES AND COMPOSITES

The mean strengths of the several fibrous structures derived above represent only the average results. It is usually more meaningful to find out the actual strengths, i.e., the peak values on the stress-strain curves, and the corresponding strains as the actual breaking strains.

According to the weakest link hypothesis, the actual strength of a fiber bundle is in fact determined by its weakest constituents. Therefore, in order to find the actual strength, we have to deal with the problem of the distribution of the smallest fiber strengths. It can be proved that the peak value on a stress-strain curve of a fibrous structure is in fact the mode of the smallest fiber strengths, or the most probable value of the strength distribution. For a normal distribution as shown in either Equation (4) or (13), this most probable value σ_m can be calculated as [14]

$$\sigma_m = \bar{\sigma} - \Theta\sqrt{2\log N} + \Theta\frac{\log\log N + \log 4\pi}{2\sqrt{2\log N}} \tag{19}$$

The corresponding mean $\bar{\sigma}$ and standard deviation Θ values should be used when applied to a fiber bundle or a composite, respectively.

It is clear from this equation that in the present case the mode σ_m is always smaller than the mean value $\bar{\sigma}$. Further, by definition, there is a higher probability than those of any other alternative strength definitions that this mode σ_m is the most likely estimate of the actual strength σ; we can therefore treat it as such so that $\sigma_m = \sigma$. Additionally, it is again shown clearly in Equation (19) that it is the strength variation which reduces the strength of a fibrous structure.

More specifically, a comparison between the mode σ_p and the mean $\bar{\sigma}_p$ of fiber bundle strengths using Equations (5) and (19) is depicted in Figure 2. Since, for a given fiber scale parameter α, this ratio is found to be related only to the fiber number N besides the fiber shape parameter β, the results at three N levels are provided here. It confirms in the figure that the mode of the bundle strength is always smaller than the mean bundle strength. This discrepancy will however reduce at a higher value of either β or N.

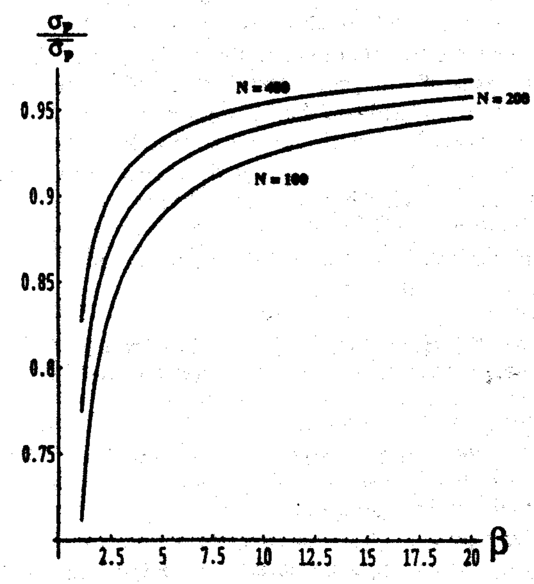


Figure 2. The effects of β and bundle size N on the $\sigma_p/\bar{\sigma}_p$ ratio.

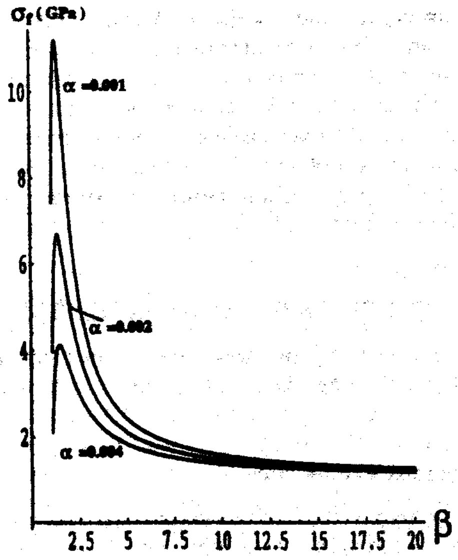


Figure 3. *Effects of β and α on the fiber strength σ_f.*

On the other hand, the most probable value of σ_{fm} of a fiber strength whose distribution function is of Weibull form is provided in Reference [14] as

$$\sigma_{fm} = \sigma_f = (n\alpha)^{-1/\beta}\left(1 - \frac{1}{\beta}\right)^{1/\beta} \tag{20}$$

where n is the number of unit length (the length of the basic links). For convenience, we choose the unit length = 1 mm so that the magnitude of n is equal to the gauge length at which fiber strength is determined. It should be pointed out now that the fiber breaking strain in Figure 1 is thus determined as σ_{fm}/E_f or σ_f/E_f.

The effects of the fiber shape and scale parameters β and α on the actual fiber strength σ_f are shown in Figure 3. There is an optimal β value for each given α at which the fiber strength will reach the maximum. This optimal condition however may not be of great practical significance since in most applications, the β values are likely above this level. We can hence conclude based on the figure that in practice when β value is not very large, increase of β level will reduce the fiber strength, and a higher α will lead to a lower fiber strength. However, as β value becomes very high, i.e., fiber strength variation reduces greatly, the fiber strength will approach a lower but more consistent value independent of both α and β levels.

To verify the difference between the actual and mean fiber strengths, the ratio $\sigma_f/\bar{\sigma}_f$ is formed based on Equations (2) and (20), and is found to be related only to β. This relationship is illustrated in Figure 4. The mode of the fiber strength as seen in the figure can be either greater or lower than the mean fiber strength, depending on the β level. So there is also a critical β value to make $\sigma_f = \bar{\sigma}_f$. It is clear though this critical value is larger than the optimal β value in Figure 3.

The results in Figures 2 and 4 may suggest that for a strength variable following normal distribution, its most probable strength cannot exceed its mean strength. While for a strength variable of Weibull nature, this strength mode can be either greater or smaller than its mean value depending upon the dispersion or the β value of the variable.

6. PREDICTIONS AND DISCUSSION

The assumed fiber and matrix properties required for calculations are provided in Table 2, and in the following discussions, unless specified otherwise, these data will be used.

6.1. Fiber Bundle versus Composite

To reveal the reinforcing mechanism in a composite, it is necessary to study the relationship between the strengths of the fiber bundle and the composite. So far we have had two strength definitions, the actual (the mode) strength and the mean strength. To investigate the difference between the strengths of the fiber bundle and the composite, we can use the strength ratios calculated based on the two strength definitions, that is, the ratio of the mean strength $\bar{\sigma}_c/\bar{\sigma}_p$ and the ratio of

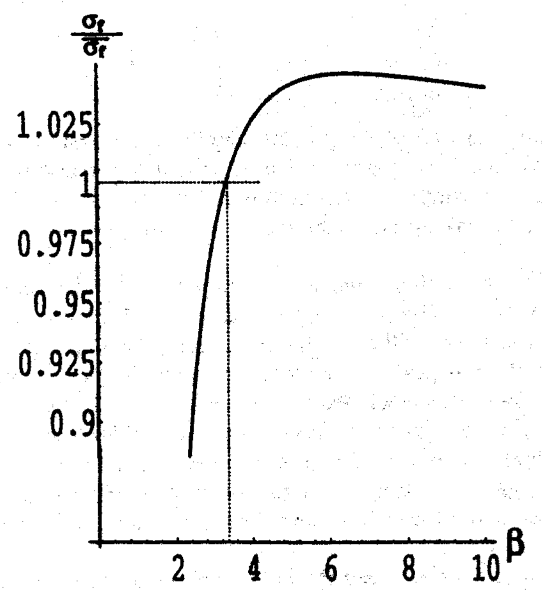


Figure 4. Effect of β on the $\sigma_f/\bar{\sigma}_f$ ratio.

the actual strength σ_c/σ_p, respectively. For easy manipulation, let us define a third ratio

$$\vartheta_{\sigma 1} = \frac{\bar{\sigma}_c}{\bar{\sigma}_p} \Big/ \frac{\sigma_c}{\sigma_p} \tag{21}$$

When $\vartheta_{\sigma 1}$ is greater or less than, or equal to one, $\bar{\sigma}_c/\bar{\sigma}_p$ is greater or less than, or equal to σ_c/σ_p. Based on Equations (5), (11) and (19) and using the data in Table 2, Figures 5(a) and (b) are plotted to show the effects of the important parameters on this ratio $\vartheta_{\sigma 1}$.

Figure 5(a) shows that the composite fiber volume fraction V_f has a significant impact on $\vartheta_{\sigma 1}$. In general, a higher V_f level leads to a greater $\vartheta_{\sigma 1}$ value. At a given β level, there is a critical V_f level beyond which $\vartheta_{\sigma 1}$ will be greater than one. The effect of the fiber shape parameter β is dependent on the value of V_f; when V_f is lower than the critical level, β has little effect except when it is very small; and once V_f exceeds the critical value, increasing the β value will lower the $\vartheta_{\sigma 1}$ result. When fiber strength variation diminishes, i.e., when $\beta \to \infty$, $\vartheta_{\sigma 1}$ will approach one, becoming almost independent of both V_f and β so that there will be $\bar{\sigma}_c/\bar{\sigma}_p = \sigma_c/\sigma_p$.

The effects of both τ_y and E_m/E_f on $\vartheta_{\sigma 1}$ value are depicted in Figure 5(b). It is seen that a higher value of either τ_y or E_m/E_f, i.e., a better interface or a stiffer matrix, will yield a greater $\vartheta_{\sigma 1}$. There also exists a critical value of τ_y, depending on the level of E_m/E_f, for $\vartheta_{\sigma 1}$ to be greater than one. Moreover, the effect of E_m/E_f on $\vartheta_{\sigma 1}$ is clearly not linear, as the three curves are spaced irregularly in the figure, although each of them corresponds to the same ten-time difference in E_m/E_f value.

It should be noted that at given levels of the related variables, $\vartheta_{\sigma 1}$ will be a constant, meaning $\bar{\sigma}_c/\bar{\sigma}_p$ is always proportional to σ_c/σ_p. Therefore, comparison between the actual strengths is equivalent in proportion to the comparison between the mean strengths of the fiber bundles and composites.

Applying Equations (5) and (11), the ratio between the mean strengths of the composites and fiber bundles, denoted as $\vartheta_{\sigma 2}$ for brevity, can be expressed as

$$\vartheta_{\sigma 2} = \frac{\bar{\sigma}_c}{\bar{\sigma}_p} = \left(V_f + \frac{V_m E_m}{E_f}\right)\left(\frac{l_c}{l_f}\right)^{-1/\beta} \tag{22}$$

And, it is easy to see from Equations (6) and (12) the strength variation ratio

$$\vartheta_{\Theta 2} = \frac{\Theta_c}{\Theta_p} = \left(V_f + \frac{V_m E_m}{E_f}\right)\left(\frac{l_c}{l_f}\right)^{-1/\beta} = \frac{\bar{\sigma}_c}{\bar{\sigma}_p} \tag{23}$$

So the discussion about Θ_c/Θ_p is equivalent to that about $\bar{\sigma}_c/\bar{\sigma}_p$.

First of all, it is clear from Equations (22) and (23) that the ratio of the critical and the original fiber lengths l_c/l_f is a key factor in determining the mean strength and the variation ratios. It is seen from Equation (8) that this length ratio is

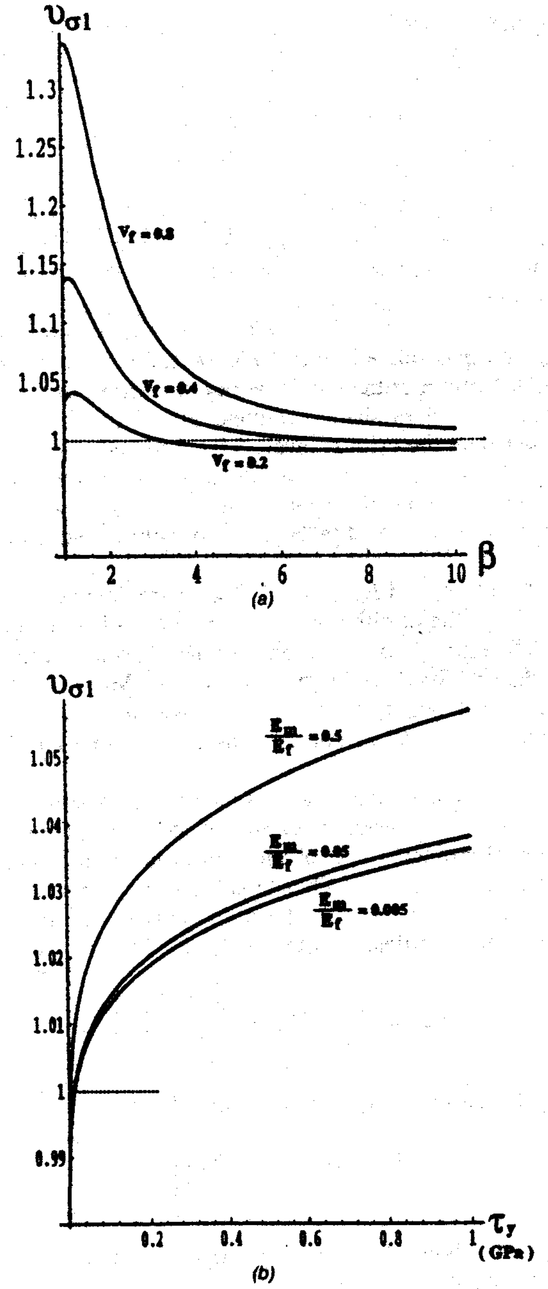


Figure 5. *Relationships between important variables and the ratio $\vartheta_{\sigma 1}$. (a) Effect of β at three V_f levels. (b) Effect of τ_y at three E_m/E_f levels.*

related to the fiber radius, the fiber shape and scale parameters as well as the interfacial parameter between fiber and the matrix. In fact, it can be readily proved that

$$\left(\frac{l_c}{l_f}\right)^{-1/\beta} = \frac{\overline{\sigma_f(l_c)}}{\overline{\sigma_f(l_f)}} \tag{24}$$

where $\overline{\sigma_f(l_c)}$ and $\overline{\sigma_f(l_f)}$ are the fiber *in situ* and the original mean strengths, calculated based on Equation (2) but using the critical length l_c defined in Equation (8) and the fiber original length l_f, respectively. The length ratio l_c/l_f is therefore an indicator of the matrix reinforcing effect on fiber strength; a smaller l_c/l_f represents a higher reinforcement to fibers from the matrix.

Figure 6 is hence plotted of the length ratio l_c/l_f against other parameters. Figure 6(a) describes the influences of α and β on the length ratio. When β is relatively small, a higher α or β value will yield a smaller l_c/l_f. But as β increases, l_c/l_f decreases to a constant regardless of the level of either parameter. The influences of τ_y and the fiber radius r_f is depicted in Figure 6(b). Generally speaking, a higher τ_y value, reflecting a better interface, generates a more significant matrix reinforcing effect on fiber and leads to a smaller l_c/l_f value. However, the effect of τ_y is more obvious when it is at a lower level. At a given τ_y level, a finer fiber will also result to a smaller l_c/l_f value.

Furthermore, it can be easily seen from Equation (22) that, theoretically, $\vartheta_{\sigma 2}$ value may not be necessarily greater than 1, meaning the strength of the composite may not be greater than that of the fiber bundle. In order for the ratio $\vartheta_{\sigma 2}$ to be greater than 1, or for a synergetic effect relative to the fiber bundle strength to occur, there are critical values for all the variables involved. In fact, based on the definition

$$\vartheta_{\sigma 2} = \frac{\bar{\sigma}_c}{\bar{\sigma}_p} = \left(V_f + \frac{V_m E_m}{E_f}\right)\left(\frac{l_c}{l_f}\right)^{-1/\beta} \geq 1 \tag{25}$$

we have the condition for the composite strength to be higher than the bundle strength as

$$V_f + (1 - V_f)\frac{E_m}{E_f} \geq \left(\frac{l_c}{l_f}\right)^{1/\beta} \tag{26}$$

First, if the fiber parameters and the interfacial property β, α, r_f and τ_y are such that the fragmentation process is prevented so that $(l_c/l_f) = 1$, the Inequality (26) will be reduced into

$$E_m \geq E_f \tag{27}$$

That is, one has to utilize a matrix stiffer than the fiber to form a composite whose mean strength is no lower than the mean strength of the fiber bundle. In

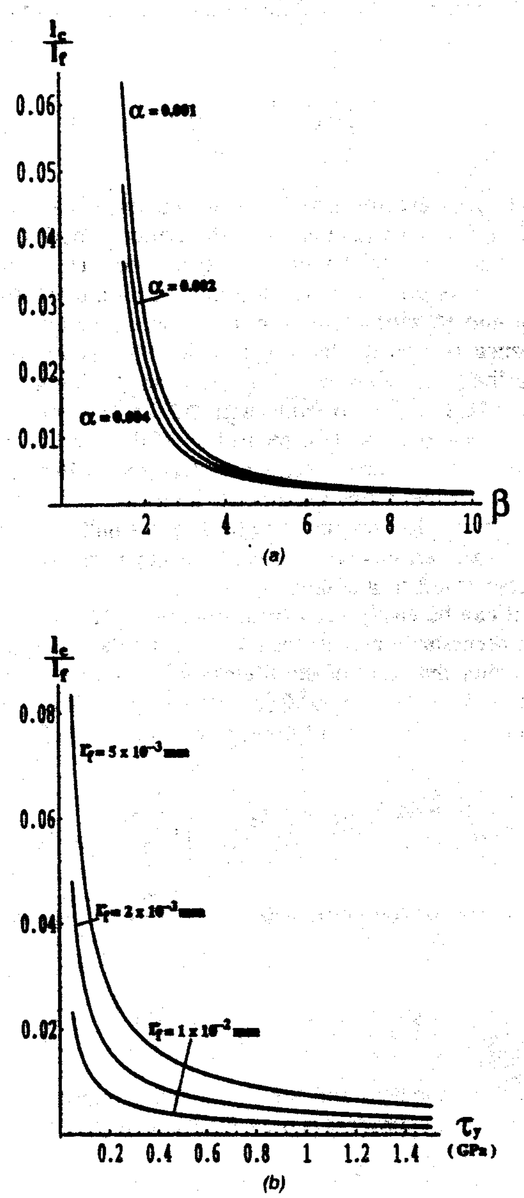


Figure 6. ***Relationships between important variables and the length ratio l_c/l_f. (a) Effect of β at three α levels. (b) Effect of τ_y at three r_f levels.***

practice, since there is usually $E_m < E_f$, the composite strength will be greater than the bundle strength only when $(l_c/l_f) < 1$, that is, only when the fragmentation process takes place. In other words, the occurrence of the fragmentation process in a composite is a necessary condition to make a composite stronger in tensile strength than the fiber bundle. However, it is shown in Figure 6 that, in the practical ranges of the related parameters, there is always $(l_c/l_f) \ll 1$.

A more detailed discussion on the ratio $\vartheta_{\sigma 2} = \bar{\sigma}_c/\bar{\sigma}_p$ is provided in Figure 7. Figures 7(a) and (b) illustrate how the variables β, α, r_f and τ_y/E_f influence the mean strength ratio. At the given parameter levels here, either decrease of β or increase of α will lead to a significant increase in $\vartheta_{\sigma 2}$ value. The effect of α however levels off very quickly, and $\vartheta_{\sigma 2}$ becomes independent of α at high level of β in Figure 7(a).

To show the function of τ_y more clearly, Figure 7(b) is plotted covering a small range of τ_y values. Most of all, the figure conforms again that a proper combination of the variables is necessary to ensure $\vartheta_{\sigma 2} > 1$. The importance of the interfacial property τ_y can be seen from the figure. When τ_y is lower than a certain level, for a given fiber radius r_f, the value of $\vartheta_{\sigma 2}$ will be smaller than 1. In other words, the synergetic effect will not occur in the case of a poor bonding between fibers and the matrix. Beyond this minimum value, a higher τ_y value, meaning a better interface, or a finer fiber results in a $\vartheta_{\sigma 2}$ ratio higher than 1.

Fiber volume fraction V_f and the tensile moduli ratio E_m/E_f also affect the $\vartheta_{\sigma 2}$ value as shown in Figure 7(c). First of all, when V_f is increasing, $\vartheta_{\sigma 2}$ value becomes larger. There is however a critical value V_{f1} beyond which $\vartheta_{\sigma 2}$ value will exceed 1 so that the synergetic effect will take place. The physical implication of

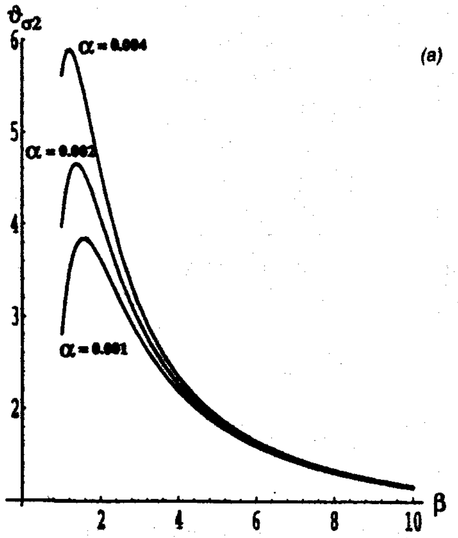
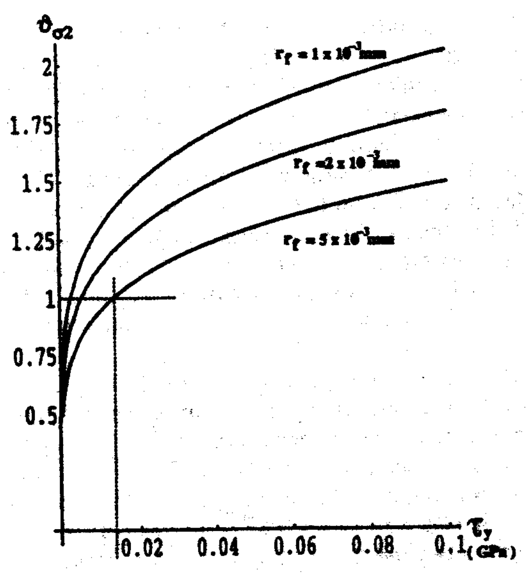
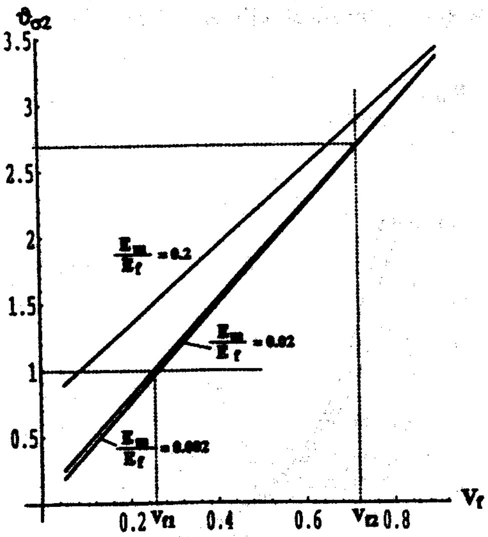


Figure 7. Relationships between important variables and the ratio $\vartheta_{\sigma 1}$. (a) Effect of β at three α levels. (b) Effect of τ_y at three r_f levels. (c) Effect of V_f at three E_m/E_f levels.



(b)



(c)

Figure 7 (continued). *Relationships between important variables and the ratio $\vartheta_{\sigma,1}$. (a) Effect of β at three α levels. (b) Effect of τ_y at three r_f levels. (c) Effect of V_f at three E_m/E_f levels.*

V_{f1} is that there has to be enough fibers, as another necessary condition, for the synergetic effect to occur. On the other hand, as proved by the present author in Reference [20] that the fiber amount in a composite cannot exceed an upper limit, indicated in Figure 7(c) as V_{f2}, to avoid the deterioration of the interfacial bonding between fiber and the matrix because of excessive fibers. So the proper range of the fiber volume fraction for the synergetic effect to realize will be $V_{f1} < V_f < V_{f2}$. In addition, although the tensile moduli ratio E_m/E_f affects the ϑ_{o2} value, the influence as seen in Figure 7(c) is not linear, and in cases where $E_m/E_f < 0.02$, the influence becomes so small as to be negligible.

6.2. Composite Strength Distribution

Next, let us examine how the important variables β, α, τ_y and V_f may alter the distribution of the composite strength. The distribution function of the composite strength is already proved to be a normal form as expressed in Equation (13). So we only need to evaluate its mean and the standard deviation. The mean value and the standard deviation of the composite strength are both related to these variables as depicted in Figure 8 using Equations (11) and (12).

Figures 8(a) and (b) show the relationships between the mean composite strength $\bar{\sigma}_c$, the strength standard deviation Θ_c, and the fiber scale parameter α and shape parameter β. Similar to Figures 7(a) and (b), when all other parameters are given, a reduction of α value will lower both the composite mean strength and its variation, and decrease of β will lead to a significant increase of both the composite mean strength and its variation, when β is small. The functions of both α and β will diminish and the composite mean strength and its variation will become independent of the two parameters at high level of β.

On the other hand, the influences of τ_y and V_f can be seen in Figures 8(c) and (d). It is easy to understand that poor bonding between fiber and matrix, represented by a small τ_y value, will lead to a lower composite mean strength as well as its variation, whereas a higher V_f value results in a higher value of both composite mean strength and its variation.

6.3. Composite Strength versus Fiber Strength

It is of more practical interest to compare the strengths of the fiber and a composite made of the fiber. This comparison is done here between their mode values as defined in Equations (19) and (20), and the results are illustrated in Figures 9(a–c) against the key parameters involved. Obviously, this strength ratio σ_c/σ_f is an indicator of the translation efficiency of fiber strength into composite strength.

It is seen from the figures that the ratio σ_c/σ_f can be either smaller or greater than 1, conforming the results shown in their stress-strain curves in Figure 1 as well as in the experimental data in Table 1. So in each figure, there is a critical level for the parameters concerned in order for the synergetic effect of composite strength in relation to fiber to occur.

Again the effects of both β and α on σ_c/σ_f ratio in Figure 9(a) are very similar to those shown in Figures 7 and 8. The critical value of β for $\sigma_c/\sigma_f > 1$ is approximately $\beta \leq 5.5$, and is barely influenced by the α level.

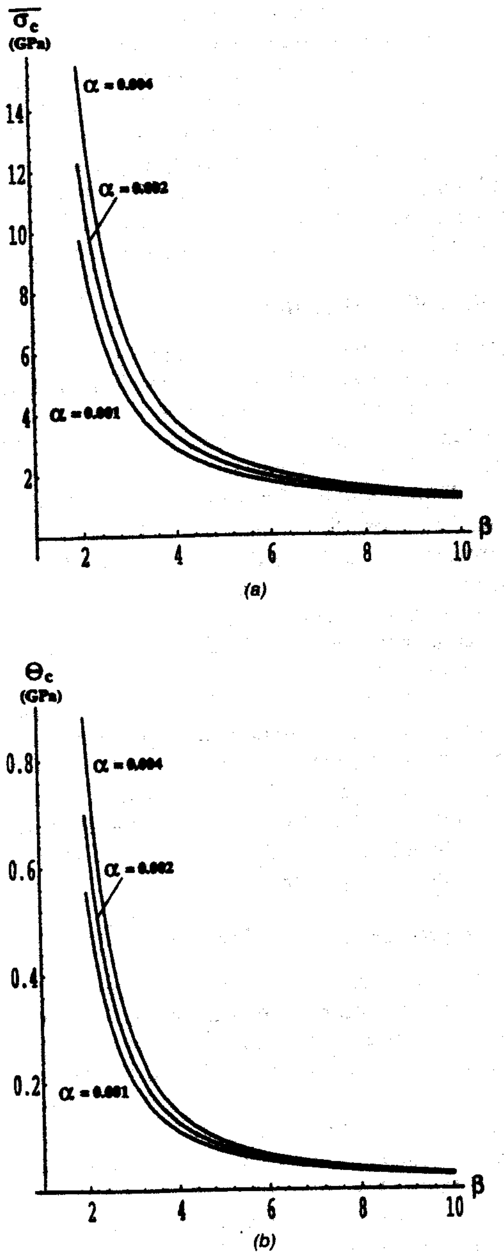
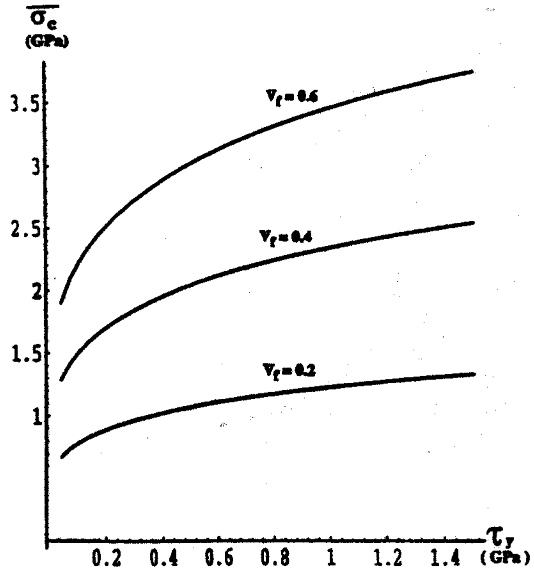
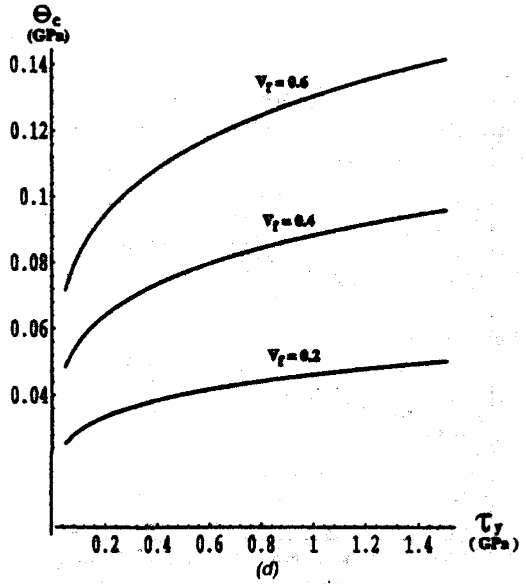


Figure 8. *Relationships between important variables and the composite strength distribution parameters. (a) Effect of β on $\bar{\sigma}_c$ at three α levels. (b) Effect of β on Θ_c at three α levels. (c) Effect of τ_y on $\bar{\sigma}_c$ at three V_f levels. (d) Effect of τ_y on Θ_c at three V_f levels.*

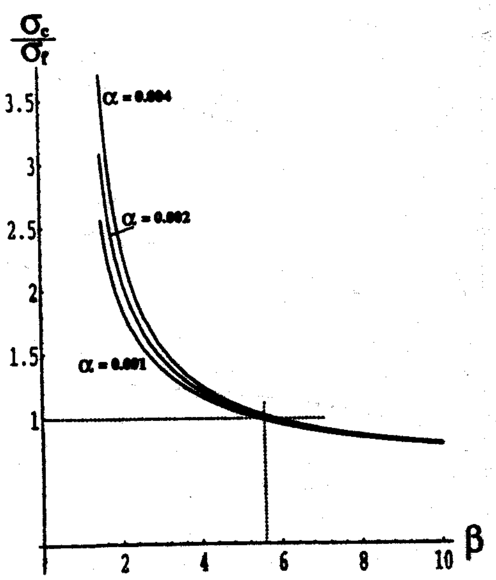


(c)

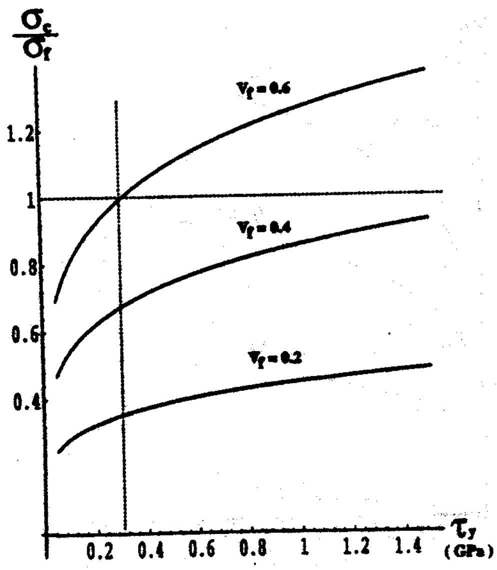


(d)

Figure 8 (continued). *Relationships between important variables and the composite strength distribution parameters. (a) Effect of β on $\bar{\sigma}_c$ at three α levels. (b) Effect of β on Θ_c at three α levels. (c) Effect of τ_y on $\bar{\sigma}_c$ at three V_f levels. (d) Effect of τ_y on Θ_c at three V_f levels.*

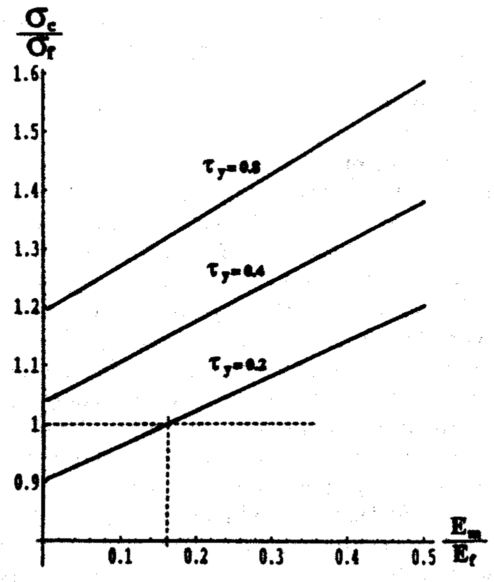


(a)



(b)

Figure 9. *Relationships between important variables and the ratio σ_c/σ_f. (a) Effect of β at three α levels. (b) Effect of τ_y at three V_f levels. (c) Effect of E_m/E_f at three τ_y levels.*



(c)

Figure 9 (continued). Relationships between important variables and the ratio σ_c/σ_f. (a) Effect of β at three α levels. (b) Effect of τ_y at three V_f levels. (c) Effect of E_m/E_f at three τ_y levels.

The bond shear yielding strength τ_y and the fiber volume fraction V_f play important roles as seen in Figure 9(b): a higher τ_y or V_f value will lead to a greater composite strength or a higher σ_c/σ_f ratio. The condition for $\sigma_c/\sigma_f > 1$ is dependent on the combination of the two parameters.

The relation between σ_c/σ_f and the modulus ratio E_m/E_f appears linear in Figure 9(c) at three τ_y levels. Once again, the condition $\sigma_c/\sigma_f > 1$ defines the critical values of the two variables for the synergetic effect in composite strength to take place.

6.4. Composite Strength Based on the Two Models

There are two theoretical models given in Equations (17) and (18) respectively in prediction of composite stress-strain relation and strength as already shown in Figure 1. The most probable strengths based on the two models can be calculated from Equations (11), (12), (17), (18) and (19), using l_f for strength σ_{cI} of model I and l_c for σ_{cII} of model II correspondingly. As mentioned before, the difference between the two strengths is caused by with or without consideration of the fiber strength variation and the fiber-matrix interactions in the composite.

It can be readily demonstrated that the ratio σ_{cII}/σ_{cI} is independent of the fiber volume fraction V_f, the modulus ratio E_m/E_f and the fiber number N, and is related only to β, α, τ_y and r_f.

First of all, the ratio σ_{cII}/σ_{cI} is proved to be always greater than 1, an indirect reflection of the synergetic effect due to the fiber-matrix interactions in a composite. There is again an optimal β value as shown in Figure 10(a) at which the ratio σ_{cII}/σ_{cI} will reach the highest level.

So far we have seen several optimal β levels in Figures 3, 4, 5(a), 7(a), 9(a) and 10(a). The optimal β level for fiber strength σ_f to be at its maximum in Figure 3 can be calculated as $\beta = 1.381$. The optimal β value associated with the mean composite strength $\bar{\sigma}_c$ (or the ratios $\vartheta_{\sigma 1}$ as well as $\vartheta_{\sigma 2}$ corresponding to Figures 5(a) and 7(a) respectively) is obtained as $\beta = 1.376$, and the identical optimal value $\beta = 1.376$ is also derived in the case of ratio σ_{cII}/σ_{cI} in Figure 10(a). The discrepancy between the above two optimal β values may suggest that the *in situ* optimal $\beta = 1.376$ is shifted (reduced) a little away from its original value $\beta = 1.381$ due to the fiber-matrix interactions in a composite. On the other hand, the critical $\beta = 5.5$ in Figure 9(a) sets the condition for composite strength to be greater than the fiber strength. Furthermore, the critical β value for fiber strength to be greater than its mean value as indicated in Figure 4 is derived as $\beta = 3.312$, but this value is of no significance to composite strength.

To summarize, the β value leading to a higher fiber strength will also result in higher mean and actual composite strengths. The *in situ* β value in a composite however will deviate slightly from the value for the fibers or fiber bundles, presumably due to the fiber-matrix interactions in the composite. When all other variables are given, β is the key factor for the composite strength synergy com-

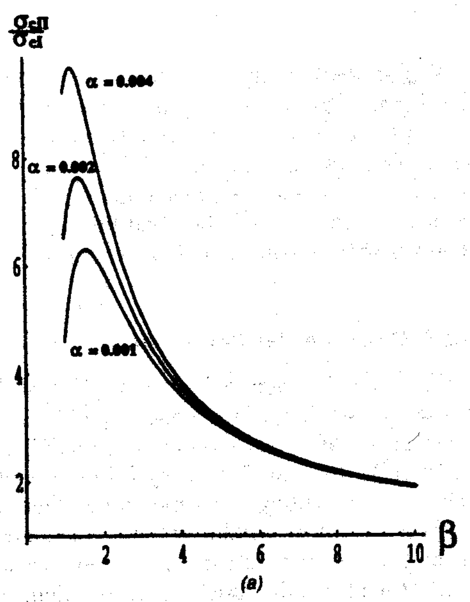
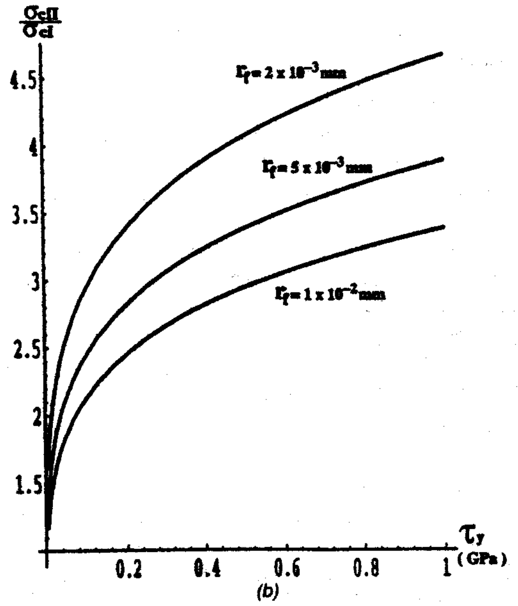


Figure 10. Relationships between important variables and the ratio σ_{cII}/σ_{cI}. (a) Effect of β at three α levels. (b) Effect of τ_y at three r_f levels.



(b)

***Figure 10 (continued).** Relationships between important variables and the ratio σ_{cII}/σ_{cI}. (a) Effect of β at three α levels. (b) Effect of τ_y at three r_f levels.*

pared with fiber strength to occur. Whether a fiber strength is greater or lower than its mean value will have no effect on the composite strength.

Furthermore, the results of σ_{cII} and σ_{cI} are compared in Figure 10(b) as functions of τ_y and r_f values. Again, a higher τ_y value or a finer fiber will increase the composite strength or the ratio σ_{cII}/σ_{cI}.

It should be noted that although the above discussions are focused on strengths, the conclusions are applicable to the breaking strains owing to the direct relation between the two.

Finally, it has to be pointed out too that, the present theory reflects the length dependence of fiber strength, in Equation (2), and of fiber bundle strength, in Equation (7). However, for a composite, as its strength is determined by the critical length l_c of the fibers as defined in Equation (8), the strength seems to be independent of the composite length. In fact as mentioned earlier, the critical length is only a representative value of the fiber fragment length that is actually a statistically distributed variable. The variation of the critical length will lead to the variation of the composite strength at different composite cross sections. Then the strength of the composite is determined by the strength of its weakest cross section which is, according to the Weakest link theory, related to the composite length. Another possible addition is the strength variation of the matrix material which may cause some size (length) effect as well.

7. CONCLUSION

The strength of fiber composite has been proved to be a normally distributed statistical variable, and its mean and variation are the functions of the mechanical properties of fibers, the matrix and the fiber-matrix interface. By including the effects associated with the fiber strength variation and the fiber-matrix interactions into the formula in calculating the parameters of the composite strength distribution function, a better model is established to yield closer prediction for the composite stress-strain curve and the composite strength.

The mode of the smallest-strength distribution is taken as an estimate for the most probable strength (MPS) of fibers, fiber bundles or composites. If the strength distribution is a normal form such as in the cases of fiber bundle and composite strengths, the MPS is always smaller than the mean strength. Whereas for a Weibull variable like fiber strength, the MPS can be either greater or smaller than its mean strength depending on the level of the fiber shape parameter β.

Compared with the simple Rule of Mixtures model, the present model predicts a much higher composite strength and a stress-strain curve much closer to the real situation.

More specifically, the actual composite strength is found to be determined by the aforementioned two mechanisms. The first is due to the fiber strength variation reflected by the fiber surviving ratio Ψ_f which leads to a span of fiber breakage and creates a round-off effect on the peak of the stress-strain curve of a composite. As a result, the composite strength and breaking strain are reduced. The second mechanism is associated with the fiber-matrix interactions during composite extension accounted for by the critical fiber length l_c. This latter reinforcing mechanism is so dominant that it can compensate for the former mechanism of the fiber strength variation, leading to an overall synergetic effect in fiber composite strength, i.e., the strength of the composite being greater than that of either the fiber bundle or the fiber.

Therefore, occurrence of the fragmentation process is a necessary condition for the synergetic effect in composite strength to take place. However, the fiber fragmentation process in a composite will occur only under a proper combination of the fiber and matrix as well as the interface properties.

To create and enhance the synergetic effect, a proper selection of the properties of the fiber and matrix is critical; finer fibers with high strength variation (small β value) and a high scale parameter α, and a stiffer matrix are desirable. When forming the composite, the fiber volume fraction should be high and the fiber-matrix interface should be strong.

Furthermore, the β value leading to a higher fiber strength will also result in higher mean and actual composite strengths. The *in situ* β value in a composite however will deviate slightly from the value for fibers and fiber bundles, presumably due to the fiber-matrix interactions in the composite. Also, β is critical for the composite strength synergy relative to fiber strength to take place. Finally, whether a fiber strength is greater or lower than its mean value will have no effect on the composite strength.

The strength-length dependence of a composite can be predicted using the present theory if we treat the critical fiber length in the composite as a statistical variable. The chance of having a weaker cross section with lower strength in a composite is then a function of the composite length.

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