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Development of a Constitutive Theory for Short Fiber Yarns

Part II: Mechanics of Staple Yarn With Slippage Effect

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ABSTRACT

A general constitutive theory governing the mechanical behavior of staple yarns and including the effect of fiber slippage during yarn extension is presented in this paper. A differential equation describing the stress transfer mechanism as well as the slippage influence in a staple yarn is applied by which both the distributions of fiber tension and lateral pressure along a fiber length are derived. A so-called fiber slippage ratio λ is defined in terms of fiber and yarn properties to specify the length along both ends of a fiber that is slipping during yarn extension. Factors such as fiber dimensions and properties, the effect of the discontinuity of fiber length within the structure, and fiber orientations in the yarn are all included in the theory. The slippage effect is incorporated into the results obtained in Part I of this study, so that the tensile and shear moduli as well as Poisson's ratios of the yarn with slippage effect are theoretically determined. Comparisons of these properties in both cases with and without the slippage mechanism are provided in this paper.

The role of twists in a continuous filament yarn is mainly to produce a coherent structure that does not readily disintegrate under lateral stress. Twist is therefore not essential in offering tensile strength to the structure, but in fact lowers the strength of a yarn because of the induction of filament obliquity [4]. However, the twists in staple fiber yarns have the primary function of binding the fibers together by friction to form a strong yarn. Twist is therefore fundamental to providing a certain minimum coherence between fibers, without which one cannot make a staple fiber yarn with significant tensile strength. This coherence depends on the frictional forces brought into play by the lateral pressures between fibers arising from the application of tensile stress along the yarn axis. The magnitude of the coherence is built up from zero at fiber ends and reaches a maximum at the middle of the fiber length, as theoretically proven in Part I [6] of this study. Because of this gradual building up of the cohesion force, in a staple fiber yarn slippage between fibers at fiber ends, where the coherence is not great enough to grip the fiber tips, will take place during yarn extension. In other words, all fibers in a staple yarn will partially slip at their ends, and will be tightly gripped at a central region along the fiber length. The length of this central region, as expected, depends on the fiber properties, the fiber orientation in the yarn, and most importantly, the twist level of the yarn.

The treatment of fiber slippage makes an already complex question more difficult. The inclusion of fiber slippage into the staple yarn model of mechanics has been the topic for several studies, and various approaches have been applied to tackle this problem. Hearle [4] investigated slippage in a staple yarn under extension by dividing all the fibers in a yarn cross section into the ones that are actively gripped by the yarn by virtue of having longer tails and those that are slipping, based on the paths or locations of the fibers within the yarn. Overall yarn properties consist of contributions from these two groups. Zurek [9], on the other hand, applied a probability density function to statistically derive the so-called active fiber lengths that are, with the same concept, the portion along the lengths of fibers gripped effectively by the yarn and therefore are able to transmit loads. Carnaby [8] used a similar probabilistic method to deal with the fiber slippage problem for a staple fiber bundle without twist. By defining a condition for gripping based on the fiber tail length distribution, he was able to calculate the strength of this fiber strand. Recently, we proposed a theory dealing with fiber slippage in a general fiber assembly under compression (Carnaby and Pan [2]). Despite all these studies, research on this problem is far from complete, and no theory has been developed to specify the stress transfer mechanism in a tensioned staple yarn where fibers are partially slipping.

In the first paper [6] in this series, we presented an attempt to develop a general constitutive theory governing the mechanical behavior of staple yarns. As the first step, we only considered the high twist case where the effect of fiber slippage during yarn extension can be ignored. By considering the yarn structures as transversely isotropic with a variable fiber-volume fraction depending on the level of twist, we were able to theoretically determine tensile and shear moduli as well as Poisson's ratios of staple yarns. We verified all these predicted results according to the constitutive restraints of continuum mechanics.

In Part II, we will try to include the effect of fiber slippage in our analysis, so as to develop a reasonably realistic yet tractable mathematical model. Starting from a single fiber, we are going to determine the relative portions of slipping and nonslipping regions on it. The average tensile stress of this fiber can then be derived. By incorporating the fiber slippage effect into a factor, we can modify all the results and equations in Part I in order to establish a new constitutive equation with a fiber slippage mechanism.

Slippage Treatment and Mean Stresses

Let us start from the differential equation used in Part I describing the stress transfer mechanism in a staple yarn. Because of the existence of slippage in fiber ends during yarn extension, the nature of the stress transfer will be different from the nonslipping case.

We again use the same Cartesian coordinate system X_1, X_2, X_3 in the staple yarn structure and the angular parameters of fiber orientation, including the base angle ϕ and the polar (or helical) angle θ as established in Part I. In addition, we set up a local coordinate system on an arbitrary fiber, as shown in Figure 1. As Hearle *et al.* [4] reported, during yarn extension, there will be slippage near both ends of all fibers and a central region along the fiber length that is gripped. In Part I we have considered a staple yarn analogous to a short fiber composite; a fiber is therefore viewed as embedded in a matrix made of neighboring fibers. Similar to the treatment on composites [7], let us assume that the slippage occurs over a length λL_f from each end of the fiber, where $L_f = \frac{l_f}{2}$, one-half the fiber length, and λ is a dimensionless parameter that depends on, besides other factors, yarn twist level. Let us name λ the fiber slippage ratio. Because of interfiber slippage, the distribution of the lateral pressure on a fiber will remain a constant g_s over this slipping portion of the fiber length λL_f . We can call g_s the yielding pressure. Over the nonslipping portion of the fiber length $L_f(1 - \lambda)$

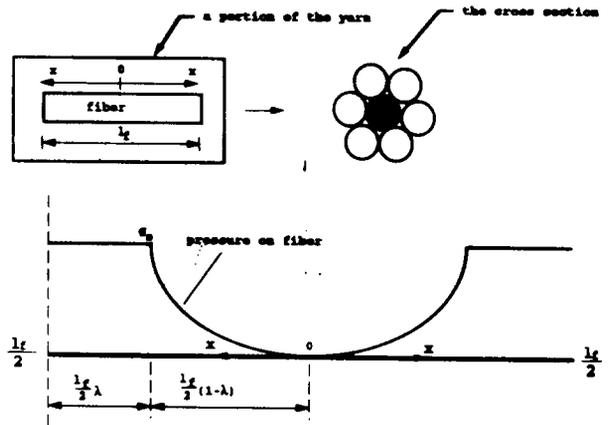


FIGURE 1. (top) A fiber embedded in a matrix made of adjacent fibers in a yarn, and (bottom) distribution of pressure on a fiber and the fiber coordinate system.

at the fiber center, the theories developed in Part I of this series for the nonslipping case should still be applicable. Therefore Equation 12 in Part I still holds for the center region as

$$\frac{d^2 C_{px}}{dx^2} = H \left(\frac{C_{px}}{E_f A_f} - \epsilon_f \right), \quad (1)$$

where C_{px} is the axial load in the fiber at a distance x from either of the two fiber ends; E_f , A_f , and ϵ_f are the fiber tensile modulus, cross sectional area, and strain; and H is a constant to be determined.

Based on the new boundary conditions reflecting fiber end slippage, this equation can be solved in a manner similar to Pigott's way of treating short fiber composites [7]:

$$\sigma_{fc} = \frac{C_{px}}{A_f} = E_f \epsilon_f - \frac{2g_s \mu \cosh(nx/r_f)}{n \sinh(n\bar{s})}, \quad (2)$$

where σ_{fc} denotes the tensile stress in the central nonslipping region of the fiber, and

$$\bar{s} = s(1 - \lambda) = \frac{l_f}{2r_f} (1 - \lambda) \quad (3)$$

can be considered as the effective fiber aspect ratio versus the original fiber aspect ratio (the fiber length-diameter ratio) $s = \frac{l_f}{2r_f}$; μ is the interfiber friction coefficient. The value n was defined as the yarn cohesion factor in Part I:

$$n = \frac{l_f}{2s} \sqrt{\frac{H}{E_f A_f}} = \sqrt{\frac{G_{TL}}{E_f} \frac{2}{\ln 2}}, \quad (4)$$

where G_{TL} is the mean longitudinal shear modulus of the yarn. The average stress in this region is thus

$$\bar{\sigma}_{fc} = \frac{1}{L_f(1-\lambda)} \int_0^{L_f(1-\lambda)} \sigma_{fc} dl = E_f \epsilon_f - \frac{2g_s \mu}{n^2 \bar{s}} \quad (5)$$

If we denote σ_{fe} as the tensile stress in the slipping portion at fiber ends, we then have, based on the force equilibrium on the fiber cross section in this region,

$$\frac{d\sigma_{fe}}{dx} = -\frac{2\mu g_s}{r_f} \quad (6)$$

or

$$\sigma_{fe} = \frac{2\mu g_s}{r_f} (L_f - x) \quad (7)$$

The average tensile stress over the fiber end is

$$\bar{\sigma}_{fe} = \frac{1}{L_f \lambda} \int_{L_f(1-\lambda)}^{L_f} \sigma_{fe} dl = \lambda s \mu g_s \quad (8)$$

The average tensile stress for the whole fiber is obtained by adding these two components weighted by their length of applied regions:

$$\bar{\sigma}_{fl} = \lambda \bar{\sigma}_{fe} + (1-\lambda) \bar{\sigma}_{fc} \quad (9)$$

The factor λ can be found using the boundary condition when $x = L_f(1-\lambda)$:

$$\sigma_{fc} = \sigma_{fe} \quad (10)$$

By solving the equation above, we then obtain

$$\lambda = \frac{E_f \epsilon_f}{2s g_s \mu} - \frac{\coth(n\bar{s})}{ns} \quad (11)$$

Substituting for $\bar{\sigma}_{fc}$ and $\bar{\sigma}_{fe}$ in Equations 5 and 8 into 9 gives the average tensile stress for the whole fiber:

$$\bar{\sigma}_{fl} = E_f \epsilon_f (1-\lambda) + \lambda^2 s \mu g_s - \frac{2\mu g_s}{n^2 \bar{s}} \quad (12)$$

We can verify this result by the following condition: When there is no slippage existing, Equation 12 should become identical to Equation 26 in Part I of the same average fiber stress but in a nonslipping case. This can be readily proved. For a nonslipping case where $\lambda = 0$, Equation 12 becomes

$$\bar{\sigma}_{fl} = E_f \epsilon_f - \frac{2\mu g_s}{n^2 \bar{s}} \quad (13)$$

Also for a nonslipping case, the yielding pressure g_s will obviously be equal to the maximum value g_{max} of the lateral pressure, which has been provided in Part I, Equation 29, as

$$g_{max} = \frac{nE_f \epsilon_f \tanh(ns)}{2\mu} \quad (14)$$

Bringing this into Equation 13 gives

$$\begin{aligned} \bar{\sigma}_{fl} &= E_f \epsilon_f - E_f \epsilon_f \frac{\tanh(ns)}{ns} \\ &= E_f \epsilon_f \left(1 - \frac{\tanh(ns)}{ns}\right) \end{aligned} \quad (15)$$

the exact result as shown in Equation 26 of Part I.

As to shear stress τ_{fc} in the central region, there exists the relation

$$\frac{d\sigma_{fc}}{dx} = -\frac{2\tau_{fc}}{r_f} \quad (16)$$

Shear stress is thus derived using Equation 2 as

$$\tau_{fc} = \mu g_c = -\frac{r_f}{2g_s} \frac{d\sigma_{fc}}{dx} = \mu \frac{g_s \sinh(nx/r_f)}{\sinh(n\bar{s})} \quad (17)$$

where g_c is the lateral pressure on the fiber at this region. Note that since λ (or \bar{s}) is a function of fiber strain, so is the shear stress τ_{fc} , although the equation above does not explicitly show this.

Since the only usable boundary condition is in Equation 10, which has been applied in determining the slippage ratio λ , the other remaining unknown parameter, the yielding or critical pressure g_s , which is related to the yielding shear stress as $\tau_y = \mu g_s$, has to be determined using other approaches. The yielding shear stress of a fibrous structure is a parameter we know little about. According to Kelly [5], for ordinary isotropic solids, this stress is proportional to the shear modulus G of the materials:

$$\tau_y = (1/25 \leftrightarrow 1/6)G \quad (18)$$

If we were to assume that this equation would still be valid for staple yarn, considering the yarn structure, we would have

$$\tau_y = (1/25 \leftrightarrow 1/6)G_{TL} \quad (19)$$

where G_{TL} is the longitudinal shear modulus of the structure.

On the other hand, we can assume a more specific form of the g_s expression by considering the following condition: For a nonslipping case, i.e., $\lambda = 0$, the maximum value of the lateral pressure at the location $x = \frac{l_f}{2}(1-\lambda)$ is, from Equation 14,

$$g_{max} \Big|_{x=(l_f/2)(1-\lambda)} = \frac{n}{2\mu} E_f \epsilon_f \tanh(n\bar{s}) \quad (20)$$

For the slippage case, however, the yielding pressure

g_s has to be different from and, most likely, lower than the result above. Let us assume that g_s is smaller by a factor,

$$\eta_{n\lambda} = \frac{1}{1 + (1 - \lambda)(\lambda n s)^2} < 1, \quad (21)$$

and becomes

$$g_s = \frac{n}{2\mu} E_f \epsilon_f \tanh(n\bar{s}) \frac{1}{1 + (1 - \lambda)(\lambda n s)^2} = \frac{n}{2\mu} E_f \epsilon_f \tanh(n\bar{s}) \eta_{n\lambda} \quad (22)$$

If we take this result into Equation 11, we will have **the solution for the slippage ratio:**

$$\lambda = \frac{\tanh(n\bar{s})}{n\bar{s}} \quad (23)$$

We can readily prove that $\lambda = 1$ when $n \rightarrow 0$, indicating that the fibers will completely slip when a yarn with zero twist is subjected to tension. Note that the physical meaning of n is the cohesion induced by the yarn twists. Equation 23 has some very interesting and important properties, which we will discuss in a later section.

The only problem with this equation is that the slippage ratio λ , as well as other system properties derived based on it, does not include the effect of the interfiber frictional coefficient μ . This is due to the fact that the yarn cohesion factor n was derived from the relationships pertinent to the nonslipping case developed in Part I. Naturally, for the nonslipping case, the frictional coefficient μ plays no roles at all. For the case with slippage effect, however, slippage ratio λ has to be related to the fiber frictional properties. This problem can be overcome by either redefining the yielding pressure g_s in Equation 22 or re-deriving the yarn cohesion factor n to include the friction mechanism. Alternatively, however, we have applied an empirical method here. Let us assume that for the slippage case, the new cohesion factor n_μ is related to the interfiber frictional coefficient μ by

$$n_\mu = \mu n \quad (24)$$

This is obviously an expedient to simplify a complex problem. However, the assumption above does lead to results that are in good agreement with the experimental evidence. We therefore need to replace n by n_μ in all the equations above.

By incorporating the expression for g_s in Equation 22 into Equation 12, we can express the average tensile stress on a fiber as

$$\begin{aligned} \bar{\sigma}_{f1} &= E_f \epsilon_f (1 - \lambda) + \lambda^2 s \mu g_s - \frac{2\mu g_s}{n_\mu^2 s} \\ &= E_f \epsilon_f \eta_{s\lambda}, \end{aligned} \quad (25)$$

where

$$\eta_{s\lambda} = \left(1 - \lambda + \frac{\tanh(n_\mu \bar{s})}{2n_\mu s} \frac{[(\lambda n_\mu s)^2 - 2]}{1 + (1 - \lambda)(\lambda n_\mu s)^2} \right) \quad (26)$$

is an efficiency factor that reflects the effects caused by fiber slippage and the definite fiber length, as well as the interaction between the two. When $n_\mu \rightarrow 0$ so that $\lambda = 1$, there will be $\eta_{s\lambda} \rightarrow 0$, meaning that no tension will be built up on fibers when yarn cohesion or twist is too low. In Part I, we have already claimed that the initial interfiber cohesion force at zero yarn twist has been ignored in the analysis. We will discuss in detail the effect of fiber slipping ends when the values of n_μ and λ are available.

Substituting Equation 25 into the previous nonslipping analysis in Part I will yield the stress-strain equation and the material constants for the yarns with fiber slippage. For the fiber orientation case where all fibers are arranged randomly in the yarn within the range defined by the yarn surface helical angle q (see details in Part I), the final results are provided below. They appear exactly the same as those in the nonslipping case in Part I, except that the factor η_1 is replaced by $\eta_{s\lambda}$.

We have the longitudinal tensile modulus of the yarn,

$$E_L = \frac{3V_f E_f \eta_{s\lambda}}{4} \frac{(1 + \cos q)^2}{1 + \cos q + \cos^2 q}, \quad (27)$$

the transverse modulus,

$$E_T = \frac{8V_f E_f \eta_{s\lambda}}{\pi^2} \times \frac{(q/2 - 1/4 \sin 2q)^2}{(2/3 - \cos q + 1/3 \cos^3 q)(1 - \cos q)}, \quad (28)$$

Poisson's ratios,

$$\nu_{LT} = \frac{\sin^5 q}{2(1 - \cos^3 q)(q/2 - 1/4 \sin 2q)}, \quad (29)$$

$$\nu_{TL} = \frac{16 \sin q (q/2 - 1/4 \sin 2q)}{3\pi^2 (2/3 - \cos q + 1/3 \cos^3 q)}, \quad (30)$$

$$\nu_{TT} = \frac{2}{\pi}, \quad (31)$$

as well as the longitudinal shear modulus,

$$G_{TL} = \frac{E_f V_f \eta_{s\lambda}}{S(T, L)}, \quad (32)$$

where

$$S(T, L) = \frac{\pi(1 - \cos q) \sin^3 q}{6(q/2 - 1/4 \sin 2q)^2} + \frac{8 \sin^3 q}{3\pi(1 - \cos q)(1 + \cos q)^2} + \frac{\pi(4 - 3 \cos q - \cos^3 q)}{6(q/2 - 1/4 \sin 2q)(1 + \cos q)}, \quad (33)$$

and the transverse shear modulus is

$$G_{TT} = \frac{4V_f E_f \eta_{s\lambda}}{\pi(2 + \pi)} \times \frac{(q/2 - 1/4 \sin 2q)^2}{(2/3 - \cos q + 1/3 \cos^3 q)(1 - \cos q)}. \quad (34)$$

Note that all Poisson's ratios here are identical to those of the nonslipping case in Part I and depend on fiber orientation only. That is, fiber slippage has no effect on them at all.

The Critical Twist Factor

Another very important issue in the analysis of staple yarn mechanics with slippage effect relates to the so-called critical yarn twist level. Hearle *et al.* have a very descriptive and detailed statement [4] about this critical yarn twist in a staple yarn. When the yarn twist level is high enough, they reported

... the stress along the fibers in a staple yarn rises from zero to the value determined by the yarn extension and then falls to zero at the other end. As twist is decreased, a point will be reached at which even midway along a fiber the tension just fails to rise to the critical value determined by the yarn extension. When this happens the self-locking features of the yarn are lost; fiber slippage becomes dominant; the fiber cannot maintain any elastic deformation; and it is impossible to build up the fiber tensions which would generate the transverse forces and in turn cause the fibers to be gripped and so allow the tension to build up.

... If all the fibers are identical in form and equivalent in path through the yarn, they will all reach the condition in which they just fail to be gripped at the same value of yarn twist. This twist will thus be a critical value at which the tension for a given extension falls sharply. Of course, in

real yarns, there will be differences in fiber dimensions, properties, and paths which will cause some fibers to slip completely while others are still gripped, leading to a blurring of the sharpness of the drop. Below the critical level of twist, the problem is one of the frictional drag of fibers sliding over one another during drafting, as discussed by Grosberg and Smith [3].

This critical twist level has its obvious theoretical and practical meanings. To start with, let us adhere to the assumptions of identical fibers and equivalent fiber paths stated above. The effect of variations of these parameters will be discussed in a later part of this series.

Derivation of this critical yarn twist level can proceed in several different approaches, depending on the criteria used. The most direct criterion is based on the condition that

$$\lambda = \frac{\tanh(n_\mu \bar{s})}{n_\mu \bar{s}} = 0. \quad (35)$$

That is, we can choose such a twist as the critical level at which fibers would be gripped entirely and slippage would be completely nonexistent. But the equation above shows that the twist level for complete gripping of fibers within the yarn is infinity, *i.e.*, when the yarn cohesion factor $n_\mu \rightarrow \infty$, indicating that fiber slippage will always take place in practice no matter how high the yarn twist level is. The yarn twist level only changes the value of λ , *i.e.*, the length of the slipping region.

It seems rational to set the critical yarn twist at such a level where all fibers will break rather than slide over each other during yarn extension. This condition can be expressed by equating the maximum tension on a fiber σ_{fm} to its ultimate strength σ_{fb} , *i.e.*,

$$\sigma_{fm} = \sigma_{fb}. \quad (36)$$

Maximum tension on a fiber can be derived from Equation 2 at $x = 0$ after substituting g_s of Equation 22 into it as

$$\begin{aligned} \sigma_{fm} &= \sigma_{fc}|_{x=0} \\ &= E_f \epsilon_f \left(1 - \frac{1}{\cosh(n_\mu \bar{s})(1 + (1 - \lambda)(\lambda n_\mu \bar{s})^2)} \right) \\ &= E_f \epsilon_f \eta_f, \quad (37) \end{aligned}$$

where we use η_f as a factor to represent the effect of fiber length and slippage. This equation indicates that as long as $\eta_f > 0$, the tension on the fiber σ_{fm} will increase along with the fiber (yarn) strain up to its breaking strength. The limiting case when this will not happen is when $\eta_f = 0$. Therefore, the equation

$$\eta_f = 1 - \frac{1}{\cosh(n_\mu \bar{s})(1 + (1 - \lambda)(\lambda n_\mu \bar{s})^2)} = 0 \quad (38)$$

will provide the solution for the critical twist level, though a solution for this equation is not readily obtainable. We have yet another criterion: the critical twist level takes the value at which the slippage ratio

$$\lambda = \frac{\tanh(n_\mu \bar{s})}{n_\mu \bar{s}} = 1 \quad (39)$$

since $\lambda = 1$ means that the slippage region extends to the entire fiber length, so that all fibers in a tensioned staple yarn are completely slipping regardless of the strain level of the yarn. In fact, both criteria in Equations 38 and 39 are equivalent, since $\eta_f = 0$ when $\lambda = 1$.

Calculation and Discussion

Now we have had all the equations describing the stress distributions on an arbitrary fiber in a tensioned yarn, the yarn properties, and the constitutive relationships including the effect of fiber slippage. A parametric study becomes possible, showing the connections between these properties and all the variables involved. The data used for calculation are listed in Table I.

TABLE I. Fiber properties used for calculation.

Item	Typical value	Unit
Fiber radius r_f	3×10^{-3}	cm
Fiber length l_f	3.0	cm
Fiber specific density ρ_f	1.31	g/cm ³
Fiber modulus E_f	6×10^7	g/cm ²
Fiber frictional coefficient μ	0.3	
Fiber aspect ratio $s = \frac{l_f}{2r_f}$	500	

The relationship between the yarn twist factor T_y and the yarn surface helical angle q was already given in Part I of this series as

$$q = \arctan \left[a_q 10^{-3} T_y \left(\frac{40\pi}{\rho_f V_f} \right)^{1/2} \right] \quad (40)$$

Because Hearle *et al.* proposed this equation [4] for filament yarns, we introduced a correction factor $a_q = 2.5$ to take into account the difference when applied to a staple yarn case. Also from Part I, we have for the yarn fiber-volume fraction at a given yarn twist level T_y ,

$$V_f = 0.7(1 - 0.78e^{-0.195T_y}) \quad (41)$$

Since the longitudinal shear modulus G_{TL} in Equation 32 depends on both the cohesion factor n_μ and the slippage ratio λ , the latter two are in turn determined by the modulus G_{TL} as shown in Equations 4 and 23. A numerical approach has to be used to solve G_{TL} , n_μ , and λ from these simultaneous equations. We must stress that the solutions of a system of these transcendental equations are difficult because the convergence of the iteration methods for such a problem is not global. So the initial estimate must be quite close to the true roots of the equations. Note also that the solutions are invariant whether we use n or n_μ for the calculation.

Once we obtain the results of n_μ , λ , q , V_f , and G_{TL} , we are able to calculate all the other parameters as shown below. First of all, let us consider the relationship between the slippage ratio λ and the yarn twist factor T_y as provided in Equation 39, where the effect of T_y is reflected through the yarn cohesion factor n_μ . Figure 2 is plotted using Equation 39 at three different fiber aspect ratio s levels. Figure 2 shows that as long as the yarn twist is lower than the critical value T_{yc} , which is determined mainly by the fiber aspect ratio s and the interfiber friction coefficient μ as shown in Equation 39, the slippage ratio λ will sustain the value of 1, indicating that fibers in a tensioned yarn will be completely slipping regardless of the level of T_y provided that $T_y < T_{yc}$. This is the period for which the self-locking mechanism has not been established, so that all fibers just simply slide over each other. When the twist level exceeds the critical twist factor T_{yc} , however,

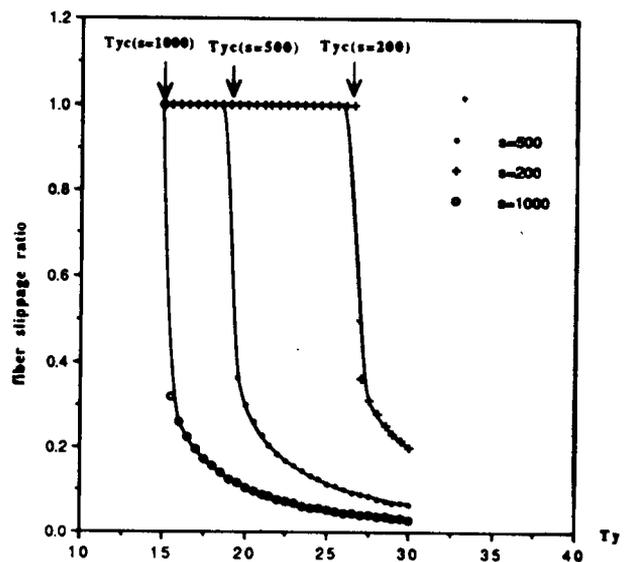


FIGURE 2. Fiber slippage ratio λ versus yarn twist factor T_y and the critical yarn twist value.

the value of λ will decrease very abruptly to a small value, and then keep decreasing at a much slower rate as the twist level increases. The tensions on yarn and fibers then start to build up. As we stated before, λ reaches zero only when $T_y \rightarrow \infty$. Therefore, in practice, the slippage at fiber ends will always take place during yarn extension no matter how high the twist level is. Higher twist levels only result in a larger portion of fiber length being gripped tightly. We must mention that the function in Equation 39 does not behave properly when the yarn twist approaches the critical twist value. We found during computation that when the yarn twist level is very close to the critical value, resonance will take place, leading to a very unstable relationship between T_y and λ . This resonance exists at an extremely narrow region around the critical twist value, and is not shown in Figure 2. This fact may indicate that at the twist level very close to critical twist, the resulting yarn will possess a structure that is very unstable with poor quality such as low strength, etc.

We should point out that the λ value predicted above is the average of all fibers. The fibers located in different radial positions in the yarn will have different slipping lengths due to the variation of lateral pressure or the yarn shear modulus. So if we can express the yarn longitudinal shear modulus G_{TL} to include its variation across yarn diameter, we will be able to show the connection between the λ value and the fiber radial position in the yarn.

Next let us examine the stress distributions on the fibers. If we replace g_s in Equation 2 with the result in Equation 22, we have the tensile stress distribution over the nonslipping portion of length $\frac{l_f}{2}(1 - \lambda)$ on each side from the fiber center as

$$\sigma_{fc} = E_f \epsilon_f \left(1 - \frac{\cosh (nx/r_f)}{\cosh (n\bar{s})} \eta_{n\lambda} \right) \quad (42)$$

For convenience, we define a unitless relative tensile stress as

$$\sigma = \frac{\sigma_{fc}}{E_f \epsilon_f} = \left(1 - \frac{\cosh (nx/r_f)}{\cosh (n\bar{s})} \eta_{n\lambda} \right) \quad (43)$$

Likewise the shear stress over the nonslipping portion becomes, from Equation 17,

$$\tau_{fc} = \mu g_c = \frac{n}{2} E_f \epsilon_f \frac{\sinh (nx/r_f)}{\cosh (n\bar{s})} \eta_{n\lambda} \quad (44)$$

Also the relative shear stress is given by

$$\tau = \frac{\tau_{fc}}{E_f \epsilon_f} = \frac{n}{2\mu} \frac{\sinh (nx/r_f)}{\cosh (n\bar{s})} \eta_{n\lambda} \quad (45)$$

Figures 3 and 4 give the distributions of both relative stresses along the fiber length at three different twist levels. For the slipping region where $x > \frac{l_f}{2}(1 - \lambda)$, the shear stress τ will keep the constant μg_s , while σ will decrease linearly down to zero at the fiber ends. Higher twist level results in flatter distributions for both stresses, and a higher rate for tensile stress to reach its maximum and for shear stress to go down to zero at the fiber center.

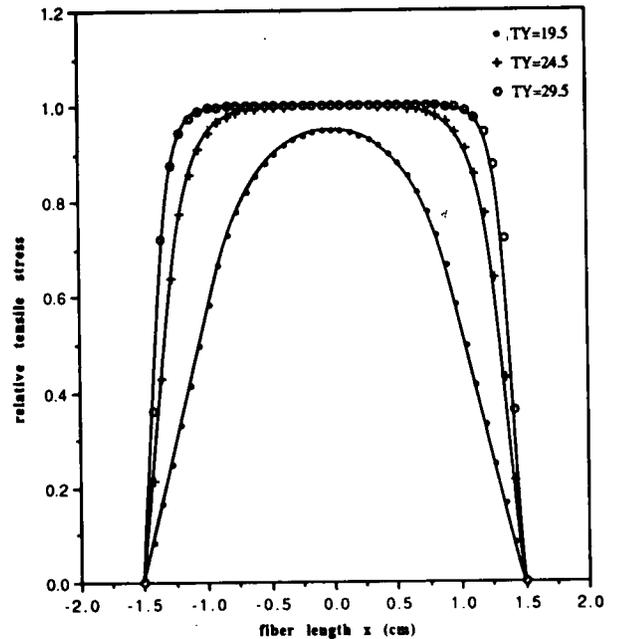


FIGURE 3. The distribution of relative tensile stress σ on a fiber.

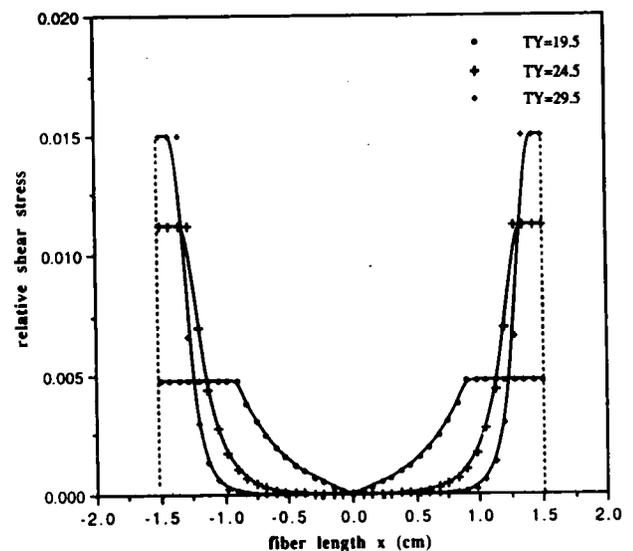


FIGURE 4. The distribution of relative shear stress τ on a fiber.

The effect of fiber slippage on yarn mechanics is represented by the differences between the factor η_l in the nonslipping case of Part I and factor η_{sl} in the slipping case. These two factors are plotted in Figure 5 against the yarn twist factor. As we stated in Part I, the factor η_l reflects the influence of limited fiber length, whereas the factor η_{sl} is the overall effect due to both fiber length and fiber slippage. Figure 5 shows that η_l starts increasing as soon as the yarn twist factor becomes greater than zero, but η_{sl} remains zero until the twist factor exceeds the critical value T_{yc} (about 20 in this example). Also η_l reaches a higher value earlier compared to η_{sl} , showing that because of fiber slippage, it will take more twist for yarn properties to reach the same values as in the nonslipping case. We will further illustrate this below.

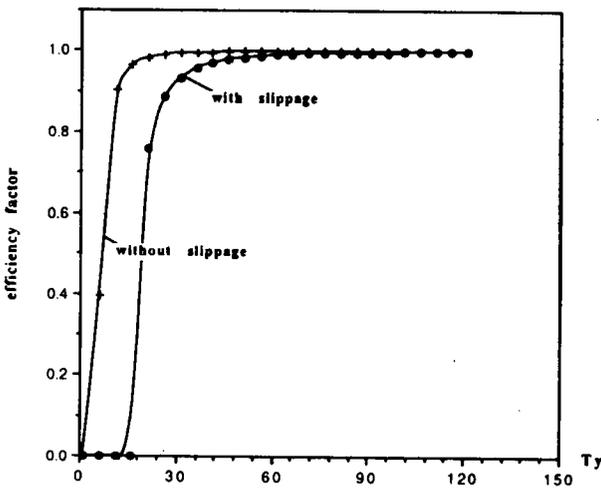


FIGURE 5. Comparison of efficiency factors with and without slippage effect.

The remaining figures provide the results of predicted yarn properties. As in Part I, since all the yarn moduli are proportional to the fiber tensile modulus E_f , the figures of yarn moduli are plotted here in terms of the relative scale using the ratio of yarn moduli and E_f .

A comparison between the relative longitudinal tensile moduli of the yarn with or without the fiber slippage effect is provided in Figure 6. Again, because of fiber slippage, the curve for the slippage case is shifted on the yarn twist axis by a value of T_{yc} . There is, of course, an optimal twist level at which the yarn modulus obtains its maximum value. This level is different for the nonslipping and slipping cases. The maximum values for both moduli are about 0.68; that is, the yarn mod-

ulus is about the 68% of the fiber modulus. This ratio will change depending on fiber orientation, maximum fiber-volume fraction of the yarn, and fiber properties. Figure 6 shows that fiber obliquity has a smaller effect in the slippage case.

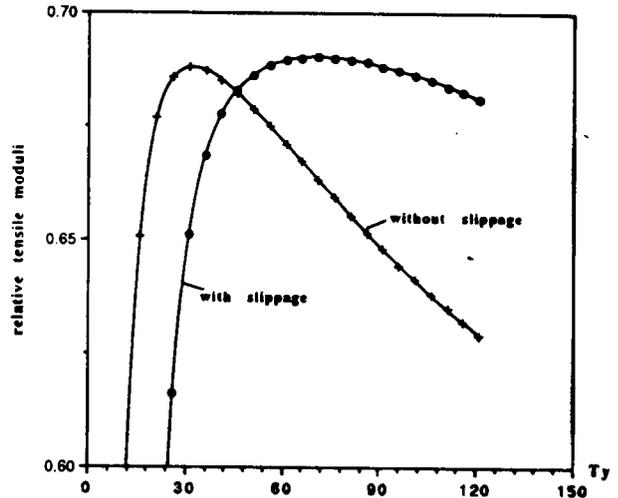


FIGURE 6. Comparison of relative longitudinal tensile moduli $\frac{E_L}{E_f}$ with and without slippage effect.

As we recognized in Part I, the fiber obliquity effect predicted here is not as significant as expected because of the form of the fiber orientation density function used in the analysis, which specifies the fiber paths in a yarn. Derivation of a more satisfactory density function will be the task in our subsequent paper focusing on an investigation of fiber orientation effects in yarn.

Figure 7 depicts the longitudinal shear moduli plotted against yarn twist factor. For a given yarn twist level, the shear modulus of the nonslipping case has a much higher value initially, and both values merge at a high twist level. Also, there is a shift on the yarn twist axis between the two curves by a value of $T_{yc} = 20$. Similar comments can be made about comparisons between the transverse tensile and shear moduli in both slipping and nonslipping cases, as shown in Figures 8 and 9, respectively. The values of these moduli provide rich information about the mechanical behavior of staple yarns.

As we mentioned earlier, Equations 29 through 31 show that the Poisson's ratios depend only on fiber orientation and therefore will possess the same values for both the slipping and nonslipping cases. The figures of these parameters plotted against the yarn twist factor were already provided in Part I. For the slippage case,

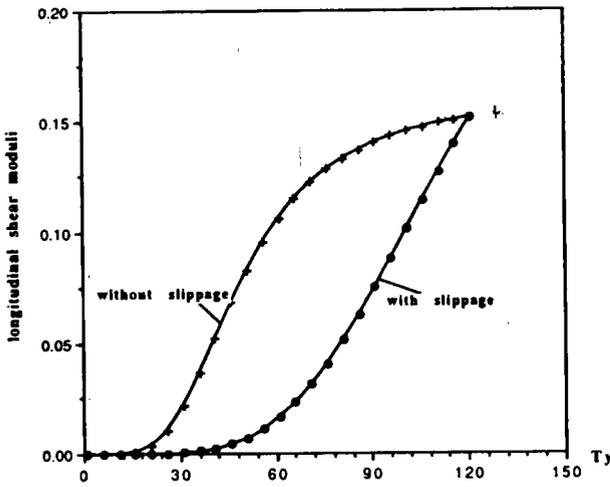


FIGURE 7. Comparison of relative longitudinal shear moduli $\frac{G_{TL}}{E_f}$ with and without slippage effect.

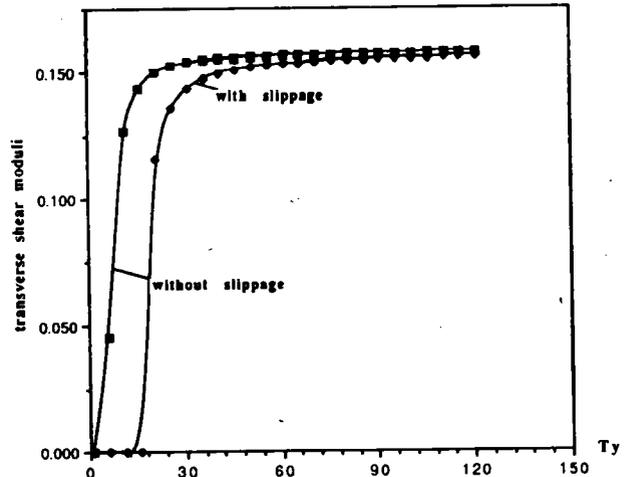


FIGURE 9. Comparison of transverse shear moduli $\frac{G_{TT}}{E_f}$ with and without slippage effect.

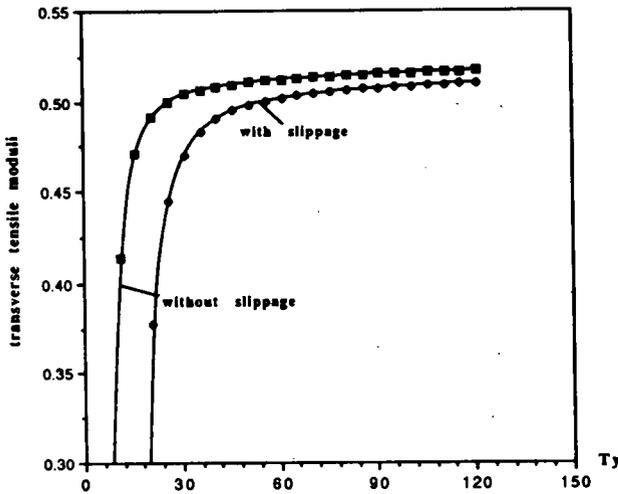


FIGURE 8. Comparison of transverse tensile moduli $\frac{E_T}{E_f}$ with and without slippage effect.

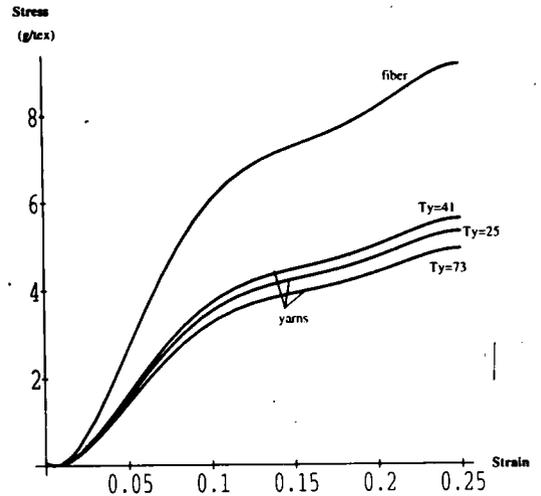


FIGURE 10. A fiber stress-strain curve and the predicted yarn stress-strain curves.

however, when the yarn twist level is lower than the critical value, no strains will develop in the fibers, so Poisson's ratios become meaningless.

In addition, it is now possible to predict in a more realistic way the stress-strain curve of a yarn once the fiber stress-strain curve is given. As stated before, we assume at this point all fibers are identical and have the same stress-strain property, as shown in Figure 10. Let us consider the uniaxial loading case where the yarn is under the axial extension only. Based on Equation 27, the stress-strain equation of the yarn can be expressed as

$$\begin{aligned} \sigma_y &= E_L \epsilon_y \\ &= \epsilon_y \frac{3V_f E_f \eta_{s\lambda}}{4} \frac{(1 + \cos q)^2}{1 + \cos q + \cos^2 q} \end{aligned} \quad (46)$$

The fiber modulus E_f equals the slope on the fiber stress-strain curve in Figure 10, depending on the level of strain in the fiber. Therefore, we need to know the connection between fiber strain ϵ_f and yarn strain ϵ_y . In general, because all fibers are located in different paths in the yarn, for the same yarn strain, the fibers will experience different strains and hence correspond to different points on the fiber stress-strain curve.

However, since yarn properties (especially the yarn modulus) are the average result of related fiber properties prior to rupture, we can use the average fiber strain for calculation. We know that if we neglect Poisson's effect, a given yarn strain ϵ_y will cause, in a fiber with helical angle θ , the strain

$$\epsilon_f = \epsilon_y \cos^2 \theta \quad (47)$$

The average fiber strain can therefore be obtained by integrating the result above using the statistical density function,

$$\bar{\epsilon}_f = \int_0^q d\theta \int_0^\pi d\phi \epsilon_f \Omega(\theta, \phi) \sin \theta \quad (48)$$

where q is the yarn surface helical angle and the density function describing fiber orientation was provided in Part I:

$$\Omega(\theta, \phi) \sin \theta = \frac{1}{\pi(1 - \cos q)} \sin \theta \quad (49)$$

Therefore, the average fiber strain is calculated as

$$\bar{\epsilon}_f = \epsilon_f \left(\frac{1 + \cos q + \cos^2 q}{3} \right) \quad (50)$$

The equation shows that when the yarn surface helical angle $q = 0$, *i.e.*, when all fibers are parallel to the yarn axis, they will experience the same fiber strain equal to the yarn strain. Based on the results in the equations above, we can predict the stress-strain curve of the yarn. The results at three different yarn twist levels are also provided in Figure 10. By comparing the curves in Figure 10, we can see that the yarn curves have a shape similar to the fiber curve, except that the slopes of the curves are different because the tensile modulus of the fiber is higher than that of the yarn. As a result, for the same strain level, the yarn specific stress is lower. The figure also shows that yarn twist changes the shape of the yarn curves. The three twist levels applied in Figure 10 also imply the existence of an optimal twist level at which yarn strength will reach a maximum.

One problem existing in the prediction above is that it does not include the effect caused by the fiber orientation change and the corresponding changes of the probability density function and the surface helix angle q during yarn extension. However, this effect is unlikely to be very significant. Carnaby and Pan [2] have dealt with the issue of changing fiber orientation density function during deformation of a fiber assembly.

The effects of fiber dimension and interfiber friction coefficient on yarn properties can be studied by examining their influence on the slippage ratio λ and the critical twist factor T_{yc} . Let us look at Equation 39 and

Figure 2 again. As defined in Equation 39, the fiber aspect ratio $s = \frac{l_f}{2r_f}$ and fiber friction coefficient μ have a similar effect on the value of the slippage ratio λ : increasing the s value is equivalent to increasing μ . In Figure 2, the results at three different s values ($s = 200, 500, \text{ and } 1000$) are equivalent to $\mu = 0.12, 0.3, \text{ and } 0.6$. So a higher s value, meaning longer or thinner fibers, or a higher friction μ will lead to a smaller critical twist level. This is in agreement with Hearle's qualitative speculation in Part I [4].

One interesting question is whether s and μ would affect the magnitudes of the yarn modulus or strength. The influence of these two parameters on yarn properties is reflected through the factor $\eta_{s\lambda}$. Since the ultimate limit for $\eta_{s\lambda}$ is 1 and the values of s and μ can only change the rate at which $\eta_{s\lambda}$ approaches the limit, but not the value of the ultimate limit, these two parameters should have no effect on the ultimate magnitudes of yarn modulus and strength. Nevertheless, in practice, when the yarn twist level is not very high, the yarn properties would seem to depend on the fiber dimensions s and friction property μ .

Furthermore, the s and μ values at the critical yarn twist factor can, in turn, be considered as the critical values for s and μ . Apparently, if we alter either value, the critical twist factor will change accordingly. In other words, for a given yarn twist factor, the structural properties of the yarn can change remarkably, since the value of s or μ alone can make the yarn twist level move beyond or below the critical twist value. A study reported recently [1] on the importance of interfiber friction on yarn strength provides experimental evidence supporting this conclusion.

Conclusions

A constitutive relationship for staple fiber yarns has been developed in this study, taking into consideration the effects of fiber slippage. We have shown theoretically that along a fiber length, there is a nonslipping portion in the center where the fiber tension and shear stress distributions obey the equations derived in Part I, and there are slipping fiber ends where the stresses are distributed in a different manner, as shown in this study. The relative ratio of these two portions can be represented by a slippage ratio λ defined in this paper. This is an important factor, depending mainly on the fiber aspect ratio s , the yarn cohesion factor n , and the interfiber friction coefficient μ .

Fiber slippage during yarn extension affects the yarn tensile and shear moduli. These parameters are proportional to the fiber tensile modulus E_f , with factors

consisting of three parts, the fiber-volume fraction V_f , the efficiency factor $\eta_{s\lambda}$ in which the fiber dimensions and slippage effects are included, and the effect of fiber obliquity or fiber orientation distribution within the yarn. Yarn twists alter the values of the yarn moduli through these factors. On the other hand, yarn Poisson's ratios are related only to the geometrical orientation of the fibers within the yarn, and are independent of the intrinsic properties and slippage effects of the fibers. Finally, we have found that critical values exist for the yarn twist factor T_y , the fiber aspect ratio s , and the interfiber frictional coefficient μ , which all have profound effects on the mechanical behavior of staple yarns.

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