A Modified Analysis of the Microstructural Characteristics of General Fiber Assemblies

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ABSTRACT

A critical problem in the study by Komori and Makishima about the microstructural characterization of general fiber assemblies is analyzed, and a modified approach is presented in this paper. The modified theory is able to predict satisfactorily such parameters as the mean values of the number of fiber contacts and fiber segments between contacts in an unbonded fiber system. For a bonded fibrous structure like a nonwoven material, the relative proportions, the distributions of the bond lengths, and the free fiber lengths are also provided based on the modified theory. The theory has been applied to three typical fibrous systems to calculate the microstructural parameters that are believed useful for further studies of these systems.

Most textile products such as clothing materials, nonwovens, and fibrous hygienic items can be considered assemblies made of fibers and air filling in the pores formed by the fibers. If we treat the products as two-component systems of fibers and air, then the properties of the systems are entirely determined by the relative amount (concentration) and distribution of the constituents, in addition to the properties of both fiber and air. We use the fiber volume fraction V_f and the fiber orientation density function $\Omega(\theta, \phi)$ to specify the two system parameters. More specifically, if we look at a typical structural element in a fibrous system, as shown in Figure 1, it consists of three portions, *i.e.*, the contact (or, when appropriate, the bonded) points between fibers, the fiber lengths between the contact points, and the void space enclosed by the adjacent fibers. Obviously, the dimensions of these microstructural characteristics are related to the fiber size, volume fraction, and orientation density function. For studies of any properties of the fibrous system, it is fundamental first to quantitatively investigate these microstructural features.

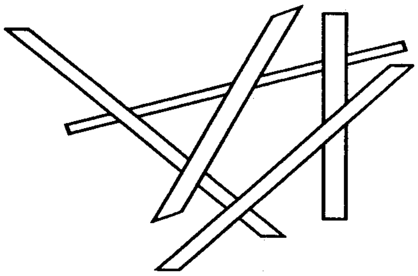

FIGURE 1. A typical structural unit in a fiber assembly.

Although extensive work has been done in paper microstructure analysis by Kallmes and his colleagues [3, 4], Page *et al.* [10, 11], and Perkins *et al.* [15, 16], Van Wyk [17] was probably among the first to study the mechanical properties of a textile fiber mass by looking into the microstructural units in the system. More complete work in this aspect, however, was done by Komori and Makishima. Through a series of papers, they predicted the mean number of fiber contact points and the mean fiber lengths between contacts [5], the fiber orientation [6], and the pore size distribution [7] of the fiber assemblies. Their results have broadened our understanding of the fibrous system and provided new means for further research work on the properties of fibrous assemblies. Several papers have since followed, based on their results for fiber assemblies: compressional properties [8], compressional hysteresis [1], and shear properties [13], as well as paper mechanics [15] and nonwoven structure behavior [14].

During the application of the Komori and Makishima's theory, however, some problems have been revealed. First, the prediction of the compressional modulus using these results was higher than the experimental data [1]. Lee *et al.* [9] then reported that there must be some problems associated with Komori and Makishima's analysis, since according to their results, the sum of all the basic volume units in a system of volume V results in a total volume smaller than V. In fact, it can be proven, as we will show in the following sections, that Komori and Makishima's prediction of the number of fiber contact points is too high, leading to a much shorter mean fiber length value. Since these issues are critical to further studies, we consider a modified approach necessary and so have proposed one in this study.

In this paper, we first demonstrate the problem and its cause in Komori and Makishima's work, and then we present a modified analysis, the new results of which are verified through several approaches. Finally, we apply the new theory to three specific cases close to real fibrous structures to provide predictions in characterizing the microstructures of these systems.

Parameters and the Problems

According to the approach explored by Komori and Makishima [5], let us first set a Cartesian coordinate system X_1, X_2, X_3 in a fibrous structure, and let the angle between the X_3 axis and the axis of an arbitrary fiber be θ, and that between the X_1 axis and the normal projection of the fiber axis onto the X_1X_2 plane be ϕ. Then the orientation of any fiber can be defined uniquely by a pair (θ, ϕ), provided that $0 \le \theta \le \pi$ and $0 \le \phi \le \pi$, as shown in Figure 2.

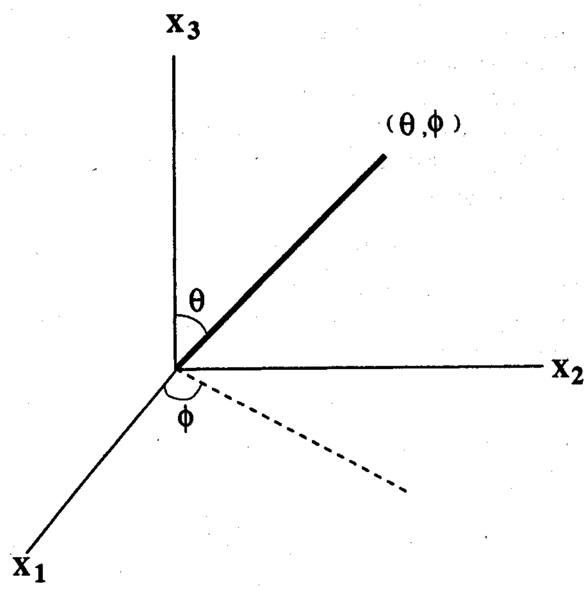

FIGURE 2. Cartesian coordinate system and a fiber in the system.

Suppose the probability of finding the orientation of a fiber in the infinitesimal range of angles $\theta \sim \theta + d\theta$ and $\phi \sim \phi + d\phi$ is $\Omega(\theta, \phi) \sin \theta d\theta d\phi$, where $\Omega(\theta, \phi)$ is the still unknown density function of fiber orientation and $\sin \theta$ is the Jacobian of the vector of the direction cosines corresponding to θ and ϕ. The following normalization condition must be satisfied:

$$\int_0^\pi d\theta \int_0^\pi d\phi \Omega(\theta, \phi) \sin \theta = 1 \quad . \tag{1}$$

Assume there are N fibers of straight cylinders of diameter $D = 2r_f$ and length l_f in the fibrous system of volume V. According to the analysis by Komori and Makishima [5], the average number of contacts on an arbitrary fiber $\bar{n}$ can be expressed as

$$\bar{n} = \frac{2DNl_f^2}{V} I \quad , \tag{2}$$

where I is a factor reflecting the fiber orientation and is defined as

$$I = \int_0^\pi d\theta \int_0^\pi d\phi J(\theta, \phi) \Omega(\theta, \phi) \sin \theta \quad , \tag{3}$$

where

$$J(\theta, \phi) = \int_0^\pi d\theta' \int_0^\pi d\phi' \Omega(\theta', \phi') \sin \chi(\theta, \phi, \theta', \phi') \sin \theta' \quad , \tag{4}$$

and

$$\sin \chi = [1 - (\cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\phi - \phi'))^2]^{1/2} \tag{5}$$

is the angle between two arbitrary fibers. The mean number of fiber contact points per unit fiber length has been derived by them as

$$\bar{n}_l = \frac{\bar{n}}{l_f} = \frac{2DNl_f}{V} I = \frac{2DL}{V} I \quad , \tag{6}$$

where $L = Nl_f$ is the total fiber length within the volume V. This equation can be further reduced to

$$\bar{n}_l = \frac{2DL}{V} I = \frac{\pi D^2 L}{4V} \frac{8I}{\pi D} = 8I \frac{V_f}{\pi D} \quad , \tag{7}$$

where $V_f = \frac{\pi D^2 L}{4V}$ is the fiber volume fraction and is usually a given parameter. We see from the result that the parameter I can be considered as an indicator of the density of contact points. The reciprocal of $\bar{n}_l$ is the mean length $\bar{b}$ between the centers of two neighboring contact points on the fiber, as illustrated in Figure 3a, *i.e.*,

$$\bar{b} = \frac{\pi D}{8IV_f} \quad . \tag{8}$$

The total number of contacts in a fiber assembly containing N fibers is then given by

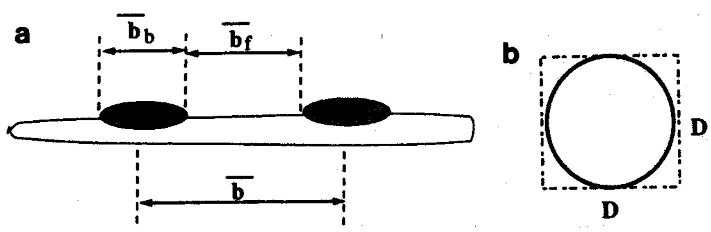

FIGURE 3. (a) Structural characteristics of a fiber element in the system. (b) Illustration for calculating the maximum fiber volume fraction.

$$n = \frac{N}{2}\bar{n} = \frac{DL^2}{V} I \quad . \tag{9}$$

Komori and Makishima introduced the factor ½ to avoid double counting of one contact. Clearly these predicted results are the basic microstructural parameters and the indispensable variables for studies of any macrostructural properties of a fibrous system. However, these predictions are subject to error. The problems existing in Komori and Makishima's analysis were demonstrated by Lee *et al.* [9], who argued that according to this method, the whole fiber assembly V can be partitioned into n units, each with volume

$$v_i = 2D\bar{b}^2 \sin \chi_i \quad . \tag{10}$$

Therefore, if the analysis were correct, the sum of all n unit volume v_i would make up the total volume V of the assembly, that is,

$$V = \sum_{i=1}^{n} v_i = \sum_{i=1}^{n} 2D\bar{b}^2 \sin \chi_i$$

$$= 2D\bar{b}^2 \sum_{i=1}^{n} \sin \chi_i \quad . \tag{11}$$

Substituting Equations 8 and 9 along with

$$\sum_{i=1}^{n} \sin \chi_i = n\langle \sin \chi \rangle \tag{12}$$

into the right side of Equation 11 yields [9]

$$2D\bar{b}^2 \sum_{i=1}^{n} \sin \chi_i = V\left(\frac{\langle \sin \chi \rangle}{2I}\right) \quad . \tag{13}$$

Lee *et al.* went on to calculate

$$\langle \sin \chi \rangle = \frac{2}{\pi}\int_0^{\pi/2} \sin \chi d\chi = \frac{2}{\pi} \quad , \tag{14}$$

so that with $I = \frac{\pi}{4}$ for a randomly oriented system, Equation 13 becomes

$$2D\bar{b}^2 \sum_{i=1}^{n} \sin \chi_i = V\left(\frac{\langle \sin \chi \rangle}{2I}\right) = V\frac{4}{\pi^2} \neq V \quad . \tag{15}$$

This provides the indication that there must be some problems associated with the theory.

We should point out, however, that Lee and his coworkers also made a mistake themselves in calculating $\langle \sin \chi \rangle$ in Equation 14. Obviously, in this equation they assumed a uniform unit density function for angle χ, which is not valid. In fact, this angle χ is a function of the fiber orientation coordinates θ, ϕ, θ', and ϕ', as defined in equation 5, whose density function is given by the expression defined in Equation 1, and the mean value of this angle therefore has to be calculated based on this density function. Note that since $\sin \chi$ is a function of the quadruple variables θ, ϕ, θ', and ϕ', its statistical mean value has to be calculated using double integration twice.

Actually, we have already shown in Equations 3 and 4 that the factor $J(\theta, \phi)$ represents just the mean value of $\sin \chi$ with respect to a given orientation (θ, ϕ) and can be designated as

$$J(\theta, \phi) = \overline{\sin \chi} \quad , \tag{16}$$

while the factor I is the overall mean value of $\sin \chi$, and is represented here by the following symbol to differentiate it from $J(\theta, \phi)$:

$$I = \langle \sin \chi \rangle \quad . \tag{17}$$

We should therefore use this result in Equation 15 to yield

$$2D\bar{b}^2 \sum_{i=1}^{n} \sin \chi_i = V\left(\frac{\langle \sin \chi \rangle}{2I}\right) = \frac{V}{2} \quad . \tag{18}$$

That is, using Komori and Makishima's conclusion, the resultant volume is only half of the assembly volume.

There is another way to detect the problem. It is easy to see that the minimum value for $\bar{b}$ should be the fiber diameter D when all fibers are regularly and fully packed together. In this case, the fiber volume fraction of the assembly will reach its maximum, which can be calculated as (see Figure 3b)

$$V_f = \frac{\pi}{4} \quad . \tag{19}$$

Also in this case, we will have $\sin \chi_i = \frac{\pi}{2}$ or $\langle \sin \chi \rangle = I = 1$. Taking these values into Equation 8 gives the value for $\bar{b}$ as

$$\bar{b} = \frac{D}{2} \ , \tag{20}$$

which is not correct. Similarly from Equation 2, we will obtain the maximum total contacts on a fiber of length l_f as

$$\bar{n} = \frac{2l_f}{D} \ , \tag{21}$$

which is also not correct because, by Komori and Makishima's definition, the maximum contacts cannot exceed $\frac{l_f}{D}$.

To conclude, based on the demonstrations provided above, there are definitely some problems in Komori and Makishima's results, which lead to erroneous predictions. A modified method is therefore necessary, and we develop one in the next section.

Modified Analysis of the Microstructure of a Fiber Assembly

According to Komori and Makishima [5], for two fibers from a fiber assembly $A(\theta, \phi)$ and $B(\theta', \phi')$ in Figure 4, the probability p of fiber B contacting fiber A in this assembly of volume V is

$$p = \frac{v(\theta, \phi; \theta', \phi')}{V} = \frac{2Dl_f^2 \sin \chi}{V} \ . \tag{22}$$

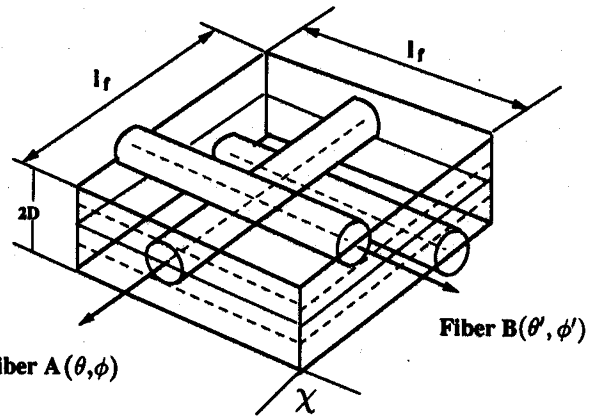

FIGURE 4. Parallelepiped formed by fiber $A(\theta, \phi)$ and $B(\theta', \phi')$ in contact [5].

Komori and Makishima claim that since there are a total of N fibers in the system, the average number of contact points of all other $N - 1$ fibers (θ', ϕ') on fiber $A(\theta, \phi)$ should be

$$n(\theta, \phi) = (N-1) \int_0^\pi d\theta' \int_0^\pi d\phi' p \Omega(\theta', \phi') \sin \theta'$$
$$= \frac{2D(N-1)l_f^2}{V} \int_0^\pi d\theta' \times \int_0^\pi d\phi' \sin \chi \Omega(\theta', \phi') \sin \theta' \ . \tag{23}$$

This is where Komori and Makishima make their mistake. They implicitly assume in this equation an independent and equal probability of making contact with fiber A for all remaining $N - 1$ fibers. This assumption is not correct, since only the first of the $N - 1$ fibers has the probability defined in Equation 22 to make contact with fiber A. For convenience, let us designate this probability of the first contact as p_1:

$$p_1 = \frac{2Dl_f^2 \sin \chi_1}{V} \ . \tag{24}$$

The probability of the second contact becomes

$$p_2 = \frac{2Dl_f^2 \sin \chi_2}{V} \left(1 - \frac{\frac{D}{\sin \chi_1}}{l_f} \right) \ , \tag{25}$$

where $\frac{D}{\sin \chi_1}$ is the fiber length occupied by the first contact point, as shown in Figure 5. That is, the successive contact probabilities are not the same as the first one, and they will be discounted by a factor, such as $\left(1 - \frac{\frac{D}{\sin \chi_1}}{l_f} \right)$ for p_2, due to the pre-existing contacts. In other words, the contact probabilities are neither the same as nor independent of each other. Likewise, the third contact corresponds to a probability:

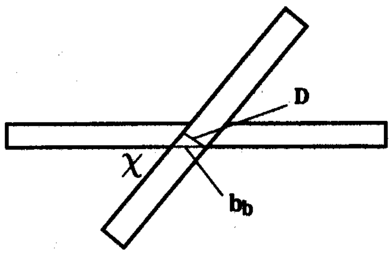

FIGURE 5. Geometry of the contact or bond portion.

$$p_3 = \frac{2Dl_f^2 \sin \chi_3}{V}\left(1 - \frac{\left(\frac{D}{\sin \chi_1} + \frac{D}{\sin \chi_2}\right)}{l_f}\right) , \quad (26)$$

and the $i + 1$th,

$$p_{i+1} = \frac{2Dl_f^2 \sin \chi_i}{V} \times \left(1 - \frac{\left(\frac{D}{\sin \chi_1} + \frac{D}{\sin \chi_2} + \ldots + \frac{D}{\sin \chi_i}\right)}{l_f}\right) . \quad (27)$$

This expression can be simplified by defining a factor $K(\theta, \phi)$ as the mean value of $\frac{1}{\sin \chi_i}$ with respect to fiber $A(\theta, \phi)$, so that

$$p_{i+1} = \frac{2Dl_f^2 \sin \chi_i}{V}\left(1 - \frac{DiK(\theta, \phi)}{l_f}\right) . \quad (28)$$

The parameter $K(\theta, \phi)$ can be calculated similar to $J(\theta, \phi)$ as

$$K(\theta, \phi) = \int_0^{\pi} d\theta' \int_0^{\pi} d\phi' \Omega(\theta', \phi') \times \frac{1}{\sin \chi(\theta, \phi, \theta', \phi')} \sin \theta' . \quad (29)$$

The average contacts of the rest of the $N - 1$ fibers with fiber $A(\theta, \phi)$ can then be calculated as the following, assuming that N is sufficiently large so that $(N - 1) \rightarrow N$:

$$n(\theta, \phi) = (N - 1)\bar{p}_{i+1} = N\bar{p}_{i+1} , \quad (30)$$

where $\bar{p}_{i+1}$ is the average contact probability and

$$\bar{p}_{i+1} = \frac{\sum_{i=1}^{n(\theta,\phi)} p_{i+1}}{n(\theta, \phi)} . \quad (31)$$

Bringing in p_{i+1} from Equation 28, the sum of the probabilities becomes

$$\sum_{i=1}^{n(\theta,\phi)} p_{i+1} = \frac{2Dl_f^2}{V} \sum_{i=1}^{n(\theta,\phi)} \left(\sin \chi_i - \frac{Di \sin \chi_i K(\theta, \phi)}{l_f}\right) = \frac{2Dl_f^2}{V}\left(n(\theta, \phi)J(\theta, \phi) - \frac{DK(\theta, \phi)}{l_f} \sum_{i=1}^{n(\theta,\phi)} i \sin \chi_i\right) . \quad (32)$$

Note that here we have used

$$\sum_{i=1}^{n(\theta,\phi)} \sin \chi_i = n(\theta, \phi)J(\theta, \phi) . \quad (33)$$

According to the statistics theory [2], for any linear combination of random variables x_i with coefficients b_i, its mean value can be expressed as

$$\langle b_i x_i \rangle = \langle B_i \rangle \langle x_i \rangle . \quad (34)$$

So we have

$$\sum_{i=1}^{n(\theta,\phi)} i \sin \chi_i = n(\theta, \phi)\langle i \sin \chi_i \rangle = n(\theta, \phi)\langle i \rangle \overline{\sin \chi_i} = n(\theta, \phi)\frac{(n(\theta, \phi) + 1)}{2} J(\theta, \phi) . \quad (35)$$

Note that according to the definition of $n(\theta, \phi)$, $J(\theta, \phi)$ has been used as the mean value of $\sin \chi_i$ with respect to fiber $A(\theta, \phi)$.

Assuming also that $n(\theta, \phi)$ is so large that $(n(\theta, \phi) + 1) \rightarrow n(\theta, \phi)$, Equation 31 can then be rewritten by incorporating Equations 32–35 into it as

$$\bar{p}_{i+1} = \frac{2Dl_f^2}{V}\left(J(\theta, \phi) - \frac{Dn(\theta, \phi)J(\theta, \phi)K(\theta, \phi)}{2l_f}\right) . \quad (36)$$

Substituting this result into Equation 30 gives

$$n(\theta, \phi) = N\frac{2Dl_f^2}{V}\left(J(\theta, \phi) - \frac{Dn(\theta, \phi)J(\theta, \phi)K(\theta, \phi)}{2l_f}\right) . \quad (37)$$

We can solve from this equation with Equation 7 that the average number of contacts on fiber $A(\theta, \phi)$ is

$$n(\theta, \phi) = \frac{8sV_fJ(\theta, \phi)}{\pi + 4V_fJ(\theta, \phi)K(\theta, \phi)} , \quad (38)$$

where $s = \frac{l_f}{D}$ is the so-called fiber aspect ratio. Then following Komori and Makishima [5], we can obtain the average number of contacts on an arbitrary fiber $\bar{n}$ by averaging $n(\theta, \phi)$ over all possible values of θ and ϕ, *i.e.*,

$$\bar{n} = \int_0^{\pi} d\theta \int_0^{\pi} d\phi n(\theta, \phi)\Omega(\theta, \phi) \sin \theta = \frac{8sV_fI}{\pi + 4V_f\Psi} , \quad (39)$$

where we have defined another function to specify the mean cross effect of $J(\theta, \phi)K(\theta, \phi)$ as

$$\Psi = \int_0^\pi d\theta \int_0^\pi d\phi J(\theta, \phi)K(\theta, \phi)\Omega(\theta, \phi)\sin\theta \quad . \quad (40)$$

Then the contacts per unit fiber length are

$$\bar{n}_l = \frac{\bar{n}}{l_f} = \frac{8sV_fI}{l_f(\pi + 4V_f\Psi)} \quad , \quad (41)$$

and the reciprocal of this gives the mean length between two contacts $\bar{b}$:

$$\bar{b} = \frac{l_f}{\bar{n}} = \frac{l_f(\pi + 4V_f\Psi)}{8sV_fI} \quad . \quad (42)$$

The total number of contacts in the assembly is

$$n = \frac{N}{2}\bar{n} = N\frac{4sV_fI}{\pi + 4V_f\Psi} \quad . \quad (43)$$

Finally, as proven by Komori and Makishima [5], the results from our theoretical analysis will not be affected by fiber crimping and the shape of fiber cross sections. However, in the case where fiber cross sections are not circular, the fiber diameter D in the results above should be replaced by the corresponding dimensions of the fiber cross section.

Verification of the Results

We can apply the same methods used before to verify these results. First of all, when fibers in the assembly are all regularly and fully packed so that the fiber volume fraction $V_f = \frac{\pi}{4}$, $\sin \chi_i = \frac{\pi}{2}$, $I = 1$, and $\Psi = 1$, from Equation 42

$$\bar{b} = D \quad . \quad (44)$$

The mean contacts on a fiber of length l_f from Equation 39 are

$$\bar{n} = s = \frac{l_f}{D} \quad (45)$$

and the total contacts in V are

$$n = \frac{Nl_f}{2D} \quad . \quad (46)$$

It is easy to see that these predictions are rational. Also, using the method proposed by Lee *et al.* [9] with Equations 12, 44, and 46, we get

$$\sum_{i=1}^{n} 2D\bar{b}^2 \sin \chi_i = 2D\bar{b}^2 n\langle \sin \chi \rangle = D^2Nl_fI = V \quad . \quad (47)$$

Note here we have used the relation

$$V_f = \frac{\pi D^2 Nl_f}{4V} = \frac{\pi}{4} \quad . \quad (48)$$

So the original volume of the assembly V is restored by adding up all the constituent units in it.

The ratio between the values of the mean fiber length $\bar{b}$ using our method and Komori and Makishima's can be calculated as

$$1 + \frac{4V_f\Psi}{\pi} > 1 \quad . \quad (49)$$

That is, Komori and Makishima's method predicts a much shorter mean fiber length between contacts; in the regularly and fully packed case as shown above, Komori and Makishima's prediction is only a half of the realistic value. In connection with this, the value of the mean contact points predicted by Komori and Makishima is much higher. This explains why the theory [1] based on these results yielded a higher system compression modulus than the experimental data.

Results for a Fiber Assembly with Bonded Fiber Contact Points

For a fibrous system where fibers just contact each other with no bond at contact points, the fiber portions where contacts take place have the same properties as the rest of the fiber. Also the contact points are not fixed, and they will move when the system is under external load. Consequently, these contacts can be treated as the ideal geometric "points" with zero area value, so that their effects on the system properties can be ignored. For a system where all fibers are bonded through various means at the contact points, however, a "new material" is formed at this bonded area, which has different properties from the fibers and thus needs to be dealt with as a distinct constituent from the fibers. In this case, we need to know not only the number of bond points and the mean fiber length between the bond points, but also the relative proportions of both the bond portion and the real free fiber length. Although we have treated this problem before (Pan *et al.* [14]), we used the results derived by Komori and Makishima then, so we believe it is desirable to briefly introduce a treatment on this issue using the modified analysis.

If we examine a typical structural element of the system in Figure 3, we readily see that the mean fiber length between two bonds $\bar{b}$ consists of a bonded portion of mean length $\bar{b}_b$ and a free fiber segment of mean length $\bar{b}_f$. That is,

$$\bar{b} = \bar{b}_f + \bar{b}_b \quad . \tag{50}$$

In Pan *et al.* [14], we defined two coefficients,

$$m_l = \frac{\bar{b}_f}{\bar{b}} \tag{51}$$

and

$$n_l = \frac{\bar{b}_b}{\bar{b}} \quad , \tag{52}$$

where

$$m_l + n_l = 1 \qquad 0 \le m_l \le 1 \text{ and } 0 \le n_l \le 1 \quad , \tag{53}$$

to represent the relative proportions of the bond portion and the free fiber length. Two extreme cases are that when $m_l = 0$ and $n_l = 1$, fibers are totally bonded together, whereas $m_l = 1$ and $n_l = 0$ represent the case when bond area doesn't exist.

Because of the different orientations of the fibers, the lengths of the bonded portions on a fiber also vary depending on the directions of all fibers involved. Consequently we have to use a procedure similar to that we used in deriving $\bar{b}$ to obtain the statistical mean values for $\bar{b}_b$ and $\bar{b}_f$. We have shown [14] that from Figure 5, there exists a bond length b_b on an arbitrary fiber,

$$b_b = \frac{D}{\sin \chi} \quad . \tag{54}$$

So the overall mean bond dimension can be expressed as

$$\bar{b}_b = DR \quad , \tag{55}$$

where

$$R = \int_0^\pi d\theta \int_0^\pi d\phi \Omega(\theta, \phi) K(\theta, \phi) \sin \theta \quad , \tag{56}$$

and the function $K(\theta, \phi)$ is already defined in Equation 29. To avoid the singular value of $\sin \chi$ in calculating $K(\theta, \phi)$ in the equation, the range of the angle χ is limited to

$$\pi - \arcsin\left(\frac{D}{l_f}\right) > \chi > \arcsin\left(\frac{D}{l_f}\right) \quad . \tag{57}$$

So once we know R, we are able to obtain the values of m_l and n_l. According to the definitions above, we have

$$m_l = \frac{\bar{b} - \bar{b}_b}{\bar{b}} \quad . \tag{58}$$

Bringing the definitions for $\bar{b}$ and $\bar{b}_b$ given in Equations 42 and 55 yields

$$m_l = 1 - \frac{8V_f IR}{\pi + 4V_f \Psi} \quad . \tag{59}$$

The value of n_l follows as

$$n_l = 1 - m_l = \frac{8V_f IR}{\pi + 4V_f \Psi} \quad . \tag{60}$$

That is, both m_l and n_l are independent of fiber diameter and length, and are determined only by the fiber volume fraction and factors related to fiber orientation.

Applications and Discussion

Here we show examples of applying the new results to structures whose fiber orientation density functions are known. As in most theoretical analysis, the structures dealt with here have been idealized so that a mathematically tractable density function can be derived.

A 3D Random System

First of all, let us apply our results to a 3D fibrous assembly where all fibers are oriented randomly with no preferred direction. According to Komori and Makishima [5], the density function for such a system is independent of θ and ϕ and is given by

$$\Omega(\theta, \phi) = \frac{1}{2\pi} \quad . \tag{61}$$

Because of the independence of θ and ϕ of all the system parameters, we have

$$J(\theta, \phi) = J(0, 0) = \int_0^\pi d\theta' \int_0^\pi d\phi' \Omega(\theta', \phi') \times \sin \chi(0, 0, \theta', \phi') \sin \theta' = \frac{\pi}{4} \quad , \tag{62}$$

where from Equation 5,

$$\sin \chi(0, 0, \theta', \phi') = [1 - \cos^2 \theta']^{1/2} = \sin \theta' \quad . \tag{63}$$

Thus,

$$I = \int_0^\pi d\theta \int_0^\pi d\phi J(0, 0) \Omega(\theta, \phi) \sin \theta = \frac{\pi}{4} \quad , \tag{64}$$

$$K(\theta, \phi) = K(0, 0) = \int_0^\pi d\theta' \int_0^\pi d\phi' \Omega(\theta', \phi') \times \frac{1}{\sin \chi(0, 0, \theta', \phi')} \sin \theta' = \frac{\pi}{2} \quad , \tag{65}$$

and

$$R = \int_0^\pi d\theta \int_0^\pi d\phi K(\theta, \phi)\Omega(\theta, \phi) \sin\theta$$
$$= \int_0^\pi d\theta \int_0^\pi d\phi K(0, 0)\Omega(\theta, \phi) \sin\theta = \frac{\pi}{2} \quad , \quad (66)$$

as well as

$$\Psi = \int_0^\pi d\theta \int_0^\pi d\phi J(\theta, \phi) K(\theta, \phi)\Omega(\theta, \phi) \sin\theta$$
$$= \frac{\pi^2}{8} \quad . \quad (67)$$

Taking all these values into the relevant equations gives us the mean contacts on an arbitrary fiber,

$$\bar{n} = \frac{8sV_f I}{\pi + 4V_f\Psi} = \frac{4sV_f}{2 + \pi V_f} \quad , \quad (68)$$

the mean contacts per unit fiber length,

$$\bar{n}_l = \frac{8sV_f I}{l_f(\pi + 4V_f\Psi)} = \frac{4V_f}{D(2 + \pi V_f)} \quad , \quad (69)$$

and the mean length between the adjacent contacts,

$$\bar{b} = \frac{l_f\pi + 4V_f\Psi}{8sV_f I} = \frac{D(2 + \pi V_f)}{4V_f} \quad . \quad (70)$$

For the bonded case the ratio of the mean free fiber length to the mean length between two bonds is

$$m_l = 1 - \frac{8V_f IR}{\pi + 4V_f\Psi} = 1 - \frac{2\pi V_f}{2 + \pi V_f} \quad , \quad (71)$$

and the ratio of the bond area becomes

$$n_l = 1 - m_l = \frac{2\pi V_f}{2 + \pi V_f} \quad . \quad (72)$$

A 2D Random System

For a planar random fiber network, the results will be different from those in a 3D system because the thickness of the present system is restricted to twice that of the fiber diameter. For this case, we need to substitute $\theta = \frac{\pi}{2}$ into our analysis in the 3D system to obtain the predictions, and note the density function has become

$$\Omega(\theta, \phi) = \frac{1}{\pi} \quad . \quad (73)$$

Because of the randomness, we obtain from Equation 4

$$J\left(\frac{\pi}{2}, \phi\right) = J\left(\frac{\pi}{2}, 0\right) = \frac{2}{\pi} \quad , \quad (74)$$

where we used

$$\sin\chi(\theta, \phi, \theta', \phi') = \sin\chi\left(\frac{\pi}{2}, 0, \frac{\pi}{2}, \phi'\right)$$
$$= [1 - \cos^2\phi']^{1/2} = \sin\phi' \quad . \quad (75)$$

We also calculated

$$I = \frac{2}{\pi} \quad . \quad (76)$$

To evaluate the factor $K\left(\frac{\pi}{2}, \phi\right)$ using Equation 29, we see that $\phi' = 0$ is now the singular point. To avoid it, we need to replace the limits of the integration by the limiting values for the range of this angle provided from Equation 57 as

$$\phi_2' > \phi' > \phi_1' \quad , \quad (77)$$

where

$$\phi_1' = \arcsin\left(\frac{D}{l_f}\right) \quad (78)$$

and

$$\phi_2' = \pi - \arcsin\left(\frac{D}{l_f}\right) \quad . \quad (79)$$

So there is

$$K\left(\frac{\pi}{2}, \phi\right) = K\left(\frac{\pi}{2}, 0\right)$$
$$= \frac{1}{\pi}\int_{\phi_1'}^{\phi_2'} d\phi' \frac{1}{\sin\phi'} = \frac{2}{\pi}A \quad , \quad (80)$$

where A is a constant related only to the fiber dimensions as

$$A = \ln\left[\cot\left(\frac{\arcsin\left(\frac{D}{l_f}\right)}{2}\right)\right] \quad . \quad (81)$$

Also from Equation 56, we calculate

$$R = \frac{2}{\pi}A \quad , \quad (82)$$

and from Equation 40,

$$\Psi = \left(\frac{2}{\pi}\right)^2 A \quad . \quad (83)$$

Taking all these values into the relevant equations gives us the mean contacts on an arbitrary fiber,

$$\bar{n} = \frac{16\pi sV_f}{\pi^3 + 16AV_f} \quad , \quad (84)$$

the mean contacts per unit fiber length,

$$\bar{n}_l = \frac{16\pi V_f}{D(\pi^3 + 16AV_f)} , \quad (85)$$

and the mean length between the two contacts,

$$\bar{b} = \frac{D(\pi^3 + 16AV_f)}{16\pi V_f} . \quad (86)$$

Figure 6 illustrates the relationship between the fiber aspect ratio s and the factor A. Factor A increases monotonically with increasing s. Therefore longer or thinner fibers will lead to fewer contact points and longer mean lengths between contacts.

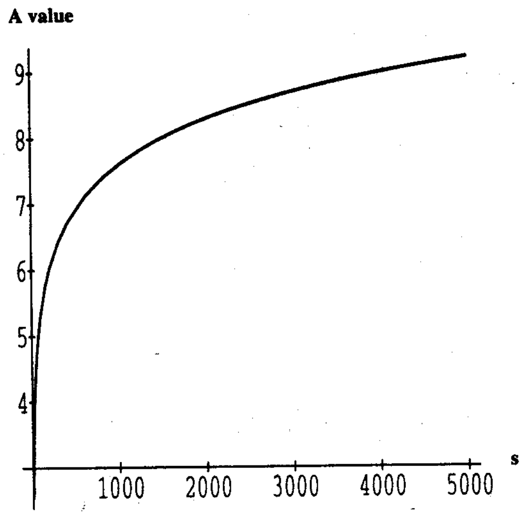

FIGURE 6. Relationship between the factor A and the fiber aspect ratio s.

For the bonded case, the ratio of the mean free fiber length to the mean length between two bonds becomes

$$m_l = 1 - \frac{32AV_f}{\pi^3 + 16AV_f} , \quad (87)$$

and the ratio of the bond length is

$$n_l = \frac{32AV_f}{\pi^3 + 16AV_f} . \quad (88)$$

A TWISTED FIBER ASSEMBLY

Most twisted fiber structures (the yarns) possess axial symmetry. Their density functions can thus be considered independent of the base angle ϕ. Moreover, note that by definition the polar angle θ is identical to the helix angle of a fiber in the yarn, so that $0 \le \theta \le q$, where q is the helix angle of fibers on the yarn surface. Since such systems are usually not bonded, our discussion excludes the bond parameters.

If we assume that all fibers in a yarn are oriented in a totally random manner within the range q, the density function becomes independent of θ as well. Therefore this density function according to our earlier paper [12] is

$$\Omega(\theta, \phi) = \frac{1}{\pi(1 - \cos q)} . \quad (89)$$

Because of the independence of the system parameters from both θ and ϕ, we again have the same expression for sin χ as given in Equation 63 for the 3D random case. So we obtain

$$J(\theta, \phi) = J(0, 0) = \int_0^q d\theta' \times \int_0^\pi d\phi' \Omega(\theta', \phi') \sin \chi(0, 0, \theta', \phi') \sin \theta' = \frac{q - 1/2 \sin 2q}{2(1 - \cos q)} , \quad (90)$$

$$I = \frac{q - 1/2 \sin 2q}{2(1 - \cos q)} , \quad (91)$$

$$K(\theta, \phi) = K(0, 0) = \frac{q}{(1 - \cos q)} , \quad (92)$$

and

$$\Psi = \frac{q(q - 1/2 \sin 2q)}{2(1 - \cos q)^2} . \quad (93)$$

Again taking all these values into Equations 41 to 43 will give us the microstructural parameters of this system.

Figure 7 shows the relationship between the mean fiber length $\bar{b}$ and the yarn surface twist angle q. For convenience, the vertical axis of the figure is plotted in a relative ratio $\frac{\bar{b}}{D}$. When the surface helical angle q increases, the length of $\bar{b}$ or the ratio will decrease. The length of $\bar{b}$ or the ratio $\frac{\bar{b}}{D}$ will be infinite if all fibers lie parallel to each other and there is no twist at all on the yarn, so that $q = 0$. Also we see from Equation 42 and the figure that the length of $\bar{b}$ or the ratio $\frac{\bar{b}}{D}$ is also affected by the fiber volume fraction. A higher V_f value will lead to a smaller value for the length of $\bar{b}$ or its ratio.

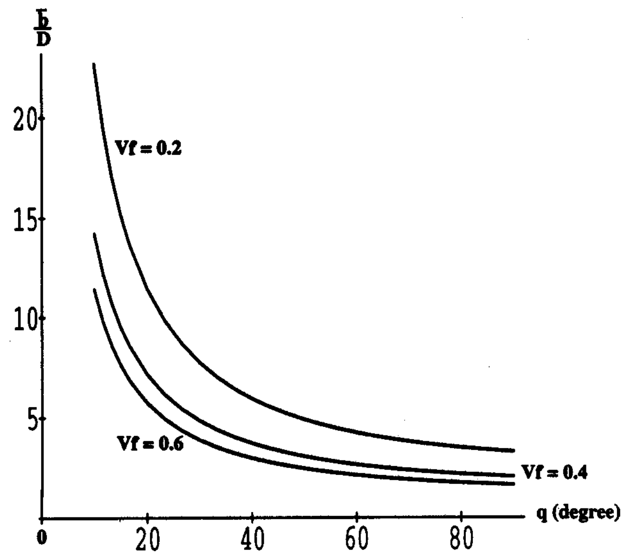

FIGURE 7. The ratio $\frac{b}{D}$ versus the surface helix angle q.

Conclusions

There is a non-negligible error in Komori and Makishima's original analysis, which leads to unrealistic predictions in characterizing fiber assemblies. The modified theory we present in this paper can be applied to calculate the microstructural characteristics of fiber assemblies, and has been verified here as rational. By means of this modified theory, the results aimed at three typical fibrous systems—perfectly random 3D and 2D structures and an ideally twisted yarn—provided in this study can be used as a basis for investigations of the various properties of practical fiber structures.

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