Reprinted from TEXTILE RESEARCH JOURNAL, Vol. 59, No. 5, May 1989
Printed in U. S. A.

Theory of the Shear Deformation of Fibrous Assemblies

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ABSTRACT

The initial response of the unit fibrous cell to an externally applied shear stress is assumed to involve both the bending of fiber sections and slippage at contact points. The criterion used to determine whether a contacting fiber will bend the contacted fiber section or whether it will slip along it depends on the relative angles of the two fiber sections to the external stresses. The total proportion of slipping and nonslipping contact points is thus derived using the density function of fiber orientations within the unit fibrous cell. The derived modulus values for shear behavior are related to the moduli derived earlier for compression behavior. The symmetry rules for selected idealized orientation distributions (*e.g.*, random) of the fibers in the unit cell provide a check on the validity of the derivations.

The use of continuum mechanics to model the behavior of assemblies of fibers remains restricted by inadequate knowledge of the constitutive properties of the unit fibrous cell. In the last three years since we first raised this issue [4], a number of papers have appeared that have improved our understanding of how the unit cell responds in tension [2] and compression [3, 8]. This understanding includes a prediction [3, 8] of the various Poisson's ratio terms so that algorithms are now available for estimating many of the tangent compliance terms in the general material properties matrix.

While these recent analyses of tension and compression have been able to draw on a long established legacy of published research dating back at least to the theories of Grosberg [5] and van Wyk [11], respectively, the same does not apply to shear behavior. Indeed while it has long been recognized that shear between layers of fibers in yarns [1] or fabrics [5] does occur, the theoretical treatment of shear as a continuum strain has received only minimal attention. Where it has been modeled, the shear deformation has been treated as frictional slippage between layers of fibers. As such there has been no consideration of the low-strain response of the assembly under shear stress, no elastic strain energy due to shear has been calculated, and only the inelastic mechanisms associated with catastrophic shear failure (*i.e.*, massive slippage) have been dealt with.

It is certainly true that such simple textile deformations as twisting a yarn [9] or bending a fabric [5] involve almost immediate shear failure, because the essentially parallel fibers slide past each other with almost no restraint in accommodating large deformations of the yarn or fabric. The simplest way of treating this problem is to assume a high initial linear shear stiffness [5] (*e.g.*, equal to the fiber modulus) up to a small critical shear stress. Thereafter, the assembly may be considered to shear via frictional slippage with no increase in shear stress. The work done against friction must be accounted for in the energy equations, but the tangent compliance in shear for initial shear stresses above the threshold level may in fact be regarded as infinite. The critical shear stress levels at which shear failure begins have been measured experimentally [4] for various initial transverse stress conditions and fiber orientations for an oriented unit cell of wool fibers.

In this paper, we present a theory of the shear mechanism for an assembly of fibers. We have assumed that the imposed shear deformation causes either slippage at contact points in the assembly or elastic bending deformation of the fiber sections between contact points. The analysis draws on a previous publication of ours [3] and follows a preliminary theoretical treatment in which no slippage at the contact points was permitted [9]. Both of these theories in turn make extensive use of earlier ideas developed by Lee and Lee [8] and Komori and Makishima [7].

0040-5175/89/59005-285$2.00

The Slippage Criterion for the General Load Case

In our earlier paper [3], we derived a general slippage condition for the unit fibrous cell under arbitrary external loading. Consider a contact point on any arbitrarily oriented fiber section whose orientation in space may be described using the polar and azimuthal angles θ and ϕ (see Figure 1). An external compressive load is applied to the assembly as a whole in the j direction such that the fiber contact point is required to sustain a contact force C_j as shown. This force C_j may be resolved into C_{jn} normal to the bottom contacting fiber and C_{jp} parallel to it.

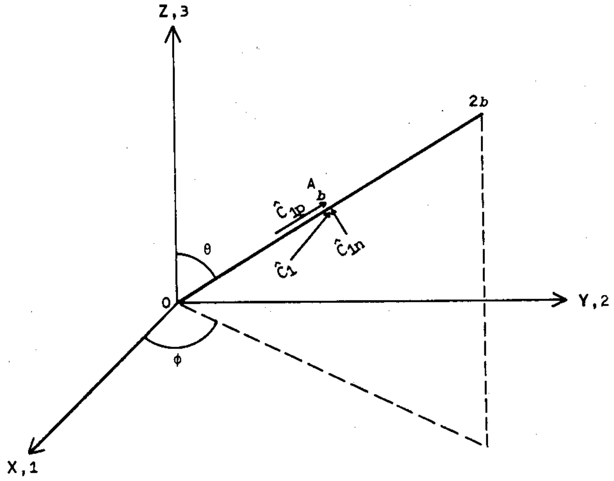

FIGURE 1. Definition of coordinate system, fiber orientation, and direction of forces.

Based on the paper of Grosberg and Smith [6] dealing with the interaction between two fibers, we proposed [3] that even in the absence of any external load on the fiber mass, the slippage of the top fiber in Figure 2 along the fiber in the direction of its axis would be opposed on average by a frictional force equal in value to WF_0b, where WF_0 is the so-called withdrawal force per unit length of fiber, which accounts for the mutual interference and frictional resistance between fiber contact points under zero external load, and b is the mean free fiber length between contact points along a fiber. The top fiber will thus begin to slide along the bottom fiber if

$$C_{jp} \geqslant C_{jn}\mu + WF_0b \quad , \tag{1}$$

where μ = coefficient of friction between two contacting fibers.

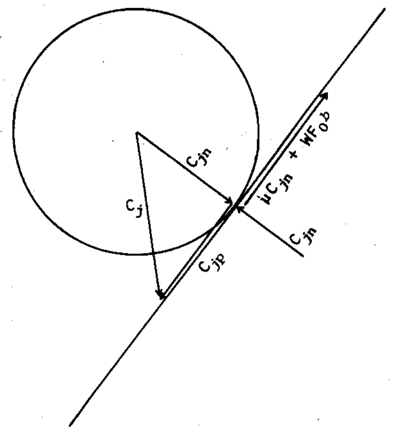

FIGURE 2. Interaction between two fibers at a contact point.

Furthermore, if the unit cell is subject to a combined compressive load (such as in a piston-and-cylinder device), the criterion for slippage given in Equation 1 remains the same but the values used for C_{jn} and C_{jp} must also include the components of force due to pressure exerted in the other orthogonal directions. Let C_1, C_2, and C_3 be the net contact forces per contact in the 1, 2, and 3 directions as shown in Figure 1. Lee and Lee [8] showed that the components of C_j normal and parallel to the arbitrary fiber section shown in Figure 1 are

$$\begin{aligned}
C_{1n} &= C_1(1 - \sin^2\theta\cos^2\phi)^{1/2} \\
C_{2n} &= C_2(1 - \sin^2\theta\sin^2\phi)^{1/2} \\
C_{3n} &= C_3\sin\theta \\
C_{1p} &= C_1\sin\theta\cos\phi \\
C_{2p} &= C_2\sin\theta\sin\phi \\
C_{3p} &= C_3\cos\theta \quad .
\end{aligned} \tag{2}$$

Equation 1 is thus equivalent for the general loading case to

$$\sum_i C_{jp} \geqslant \mu \sum_i C_{jn} + WF_0b \quad , \tag{3}$$

i.e.,

$$\begin{aligned}
&C_1\sin\theta\cos\phi + C_2\sin\theta\sin\phi + C_3\cos\theta \\
&\quad \geqslant \mu\{C_1(1 - \sin^2\theta\cos^2\phi)^{1/2} + C_2(1 - \sin^2\theta \\
&\quad\quad \times \sin^2\phi)^{1/2} + C_3\sin\theta\} + WF_0b \quad .
\end{aligned} \tag{4}$$

The slippage criterion defined in Equation 4 leads to a modified value for the parameter I, say I' (which now specifies the density of the nonslipping contact points on a fiber), as well as the length b, say b', between effective or truly supporting contact points. This modified value b', which varies with the load case in question, is used in the beam bending equation to calculate the deflection. We now realize that any numerical solution of Equation 4 will be sufficient to evaluate I' and hence b'.

Let the numerical solution of Equation 4 (for the equality only) be

$$\phi_{\text{crit}} = A(C_1, C_2, C_3, \theta) \quad . \tag{5}$$

This means that for any fiber section with orientation θ, there will only be a range of values for ϕ where slippage will not occur. If $\Omega(\theta, \phi) \sin\theta \, d\theta d\phi$ is the probability that any given fiber section lies with an orientation in the range θ to $\theta + d\theta$ and ϕ to $\phi + d\phi$, we can evaluate the double integral for I' using the numerical solution for ϕ_{crit}':

$$I' = 4 \int_0^{\pi/2} \int_{A(C_1,C_2,C_3,\theta)}^{\pi/2} J'(\theta, \phi)\Omega(\theta, \phi) \sin\theta \, d\phi d\theta \quad , \tag{6}$$

where

$$J'(\theta, \phi) = 4 \int_0^{\pi/2} \int_{A(C_1,C_2,C_3,\theta')}^{\pi/2} \Omega(\theta', \phi') \sin\theta' \times [1 - \{\cos\theta \cos\theta' + \sin\theta \sin\theta' + \cos^2(\phi - \phi')\}]^{1/2} d\phi' d\theta' \quad , \tag{7}$$

whence [3]

$$b' = V/2DLI' \quad , \tag{8}$$

where V = volume of the fiber assembly, D = fiber diameter, and L = total length of fiber in V. (Note that the integrand limits might need to be interchanged depending on the sign of $A(C_1, C_2, C_3, \theta)$ to ensure that ϕ remains in the range of 0 to $\pi/2$.)

This then provides the means of obtaining tangent compliance values for the unit cell for a wide range of initial loading conditions. In this paper, however, we will consider only the special case of simple shear stress.

Slippage Condition for Simple Shear Stress

For simple shear, there must be two forces P_j and P_k of the same magnitude acting on a fibrous unit cube in orthogonal directions so as to maintain equilibrium. Let this cube be of volume V and the length of each side therefore be $V^{1/3}$ as shown in Figure 3. Consider just one case where, with reference to Figure 3, surface tractions P_j ($j = 3$) and P_k ($k = 1$) are applied. These tractions result in as yet undetermined net point contact forces C_j and C_k at any arbitrary contact point under consideration. The slippage criterion in Equation 4 thus becomes

$$C_3 \cos\theta + C_1 \sin\theta \cos\phi \geqslant \mu\{C_3 \sin\theta + C_1(1 - \sin^2\theta \cos^2\phi)^{1/2}\} + WF_0 b \quad . \tag{9}$$

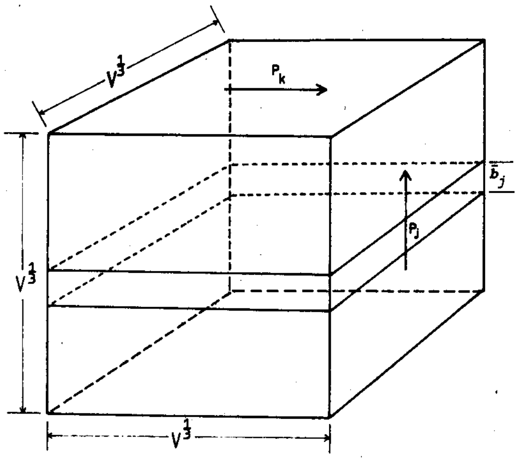

FIGURE 3. Unit cube of volume V.

Rearrangement of Equation 9 gives an explicit expression for ϕ in terms of C_3, C_1, μ, WF_0, b, and θ, which can be solved by substitution to yield a critical value $\phi_{\text{crit}}(C_3, C_1, \theta)$ for any value of θ, providing the values of the other terms in the expression are known. Thus b' is obtained from Equation 8, using $\phi_{\text{crit}}(C_3, C_1, \theta)$ to calculate J' and I' from Equations 7 and 6.

The mean projections of b' in three directions are given [3] by

$$\bar{b}_j' = 2VK_j/DLI' \quad , \tag{10}$$

where [8]

$$K_1 = \int_0^{\pi/2} d\phi \int_0^{\pi/2} d\theta \sin^2\theta \cos\phi \, \Omega(\theta, \phi) \quad , \tag{11}$$

$$K_2 = \int_0^{\pi/2} d\phi \int_0^{\pi/2} d\theta \sin^2\theta \sin\phi \, \Omega(\theta, \phi) \quad , \tag{12}$$

and

$$K_3 = \int_0^{\pi/2} d\phi \int_0^{\pi/2} d\theta \sin\theta \cos\theta \, \Omega(\theta, \phi) \quad . \tag{13}$$

Forces at Contact Points

Following an argument similar to that used in our previous treatment [3], consider all the slipping and nonslipping contact points n_{bj} in a small volume of height $\bar{b}_j'$ and cross-sectional area $V^{2/3}$ in the cube V. If SNG is the proportion of contact points that slip, then the total external surface tractions may be expressed in terms of the contributions of both slipping and nonslipping parts as

$$P_j = SNG n_{bk} C_{sj} + (1 - SNG) n_{bk} C_j \quad , \qquad (14)$$

and

$$P_k = SNG n_{bj} C_{sk} + (1 - SNG) n_{bj} C_k \quad , \qquad (15)$$

where C_j or C_k = average force per nonslipping contact point;

$$SNG = 4 \int_0^{\pi/2} \int_0^{\phi_{crit}} \Omega(\theta, \phi) \sin\theta d\theta d\phi \quad , \qquad (16)$$

$$n_{bj} = 2LK_j/V [8] \quad , \qquad (17)$$

and C_{sk} or C_{sj} = mean value of the resistance per slipping point, which can be found using the slippage criterion, Equation 9:

$$C_{sj} = 4 \int_0^{\pi/2} \int_0^{\phi_{crit}} \Omega(\theta, \phi) \sin\theta \times \frac{WF_0 b - C_k\{\sin\theta\cos\phi - \mu(1 - \sin^2\theta\cos^2\phi)^{1/2}\}}{\cos\theta - \mu\sin\theta} d\theta d\phi \quad , \qquad (18)$$

and

$$C_{sk} = 4 \int_0^{\pi/2} \int_0^{\phi_{crit}} \Omega(\theta, \phi) \sin\theta \times \frac{WF_0 b - C_j(\cos\theta - \mu\sin\theta)}{\sin\theta\cos\phi - \mu(1 - \sin^2\theta\cos^2\phi)^{1/2}} d\theta d\phi \quad . \qquad (19)$$

Note that because P_k (P_j) is a shear force, we must use $n_{bj}(n_{bk})$, not $n_{bk}(n_{bj})$, to relate C_k (C_j) to P_k (P_j) in Equation 15 (14).

Based on the shear stress equilibrium principle, the shear stress is

$$|\tau_{jk}| = |P_k/A| = |\tau_{kj}| = |P_j/A| \quad , \qquad (20)$$

where $A = V^{2/3}$, the area of the cross section. Thus,

$$|P_k| = |P_j| \quad . \qquad (21)$$

Bringing Equations 14 and 15 into Equation 21 yields a constitutive condition between C_j and C_k:

$$C_j = \frac{-SNG}{1 - SNG} C_{sj} + \frac{SNGK_j}{1 - SNGK_k} C_{sk} + \frac{K_j}{K_k} C_k \quad . \qquad (22)$$

Shear Stress-Strain Relationship

In order to define a strain, the deformations caused by the external surface tractions P_j and P_k have to be derived. P_j and P_k are transferred internally through all the contact points. Once again the bending deformation of the fiber sections is the main mechanism of resistance to the imposed shear stress.

Both van Wyk [11] and Lee and Lee [8] made various simplifying assumptions before calculating the deflection of the fibers due to the action of the contact forces. With reference to Figure 4, we will consider a single initially straight section of one fiber that is bounded by two nonslipping contact points at a distance $2b'$ apart. Let a third contact point fall at the midpoint. If we consider a ıeference frame in which points A and C are fixed, we are interested in the deflection of the midpoint of the fiber section under the action of the midpoint contact force.

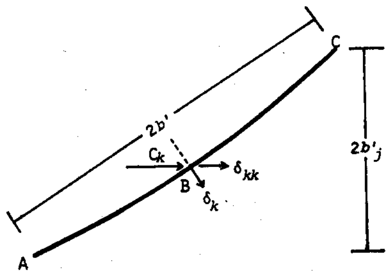

FIGURE 4. Forces acting on the fiber element.

As explained previously [3], in order to ensure continuity in curvature at the contact points, conservation of mass, and no slippage at the (nonslipping) contact points, we use van Wyk's equation [11] here to calculate the beam deflection of the fiber section. After averaging the deflection over all possible directions of the fibers, the mean deformation of the small volume of height $\bar{b}_j'$ in the cube V is derived as

$$\bar{\delta}_{jk} = \frac{1}{6} \frac{C_j b'^3}{B} m_{jk} \quad , \qquad (23)$$

where $\bar{\delta}_{jk}$ = the mean deflection of all the midpoints of the fiber sections in direction k caused by a midpoint load in direction j, and m_{jk} is as given by Lee and Lee [8]:

$$m_{11} = \int_0^{\pi/2} d\phi \int_0^{\pi/2} d\theta (1 - \sin^2\theta\cos^2\theta) \times \Omega(\theta, \phi) \sin\theta$$

$$m_{22} = \int_0^{\pi/2} d\phi \int_0^{\pi/2} d\theta (1 - \sin^2\theta \sin^2\phi) \times \Omega(\theta, \phi) \sin\theta$$

$$m_{33} = \int_0^{\pi/2} d\phi \int_0^{\pi/2} d\theta (\sin^2\theta)\Omega(\theta, \phi) \sin\theta$$

$$m_{12} = m_{21} = \int_0^{\pi/2} d\phi \int_0^{\pi/2} d\theta \sin^2\theta \sin\phi \cos\phi\, \Omega(\theta, \phi) \sin\theta$$

$$m_{23} = m_{32} = \int_0^{\pi/2} d\phi \int_0^{\pi/2} d\theta \sin\theta \cos\theta \sin\phi\, \Omega(\theta, \phi) \sin\theta$$

$$m_{31} = m_{13} = \int_0^{\pi/2} d\phi \int_0^{\pi/2} d\theta \sin\theta \cos\theta \cos\phi\, \Omega(\theta, \phi) \sin\theta \ . \tag{24}$$

The shear strain now follows by definition [10]. If α and β are the angular deformations of the face in the Z,Y plane with respect to the Z and Y axes, respectively, caused by P_k and P_j, then with reference to Figure 3,

$$\alpha \simeq \tan\alpha = (\bar{\delta}_{kk} + \bar{\delta}_{jk})/\bar{b}_j' \ , \tag{25}$$

and

$$\beta \simeq \tan\beta = (\bar{\delta}_{jj} + \bar{\delta}_{kj})/\bar{b}_k' \ . \tag{26}$$

So the shear strain due to P_j and P_k can be written as

$$\gamma_{jk} = \frac{\bar{\delta}_{kk} + \bar{\delta}_{jk}}{\bar{b}_j'} + \frac{\bar{\delta}_{jj} + \bar{\delta}_{kj}}{\bar{b}_k'} \ . \tag{27}$$

The shear stress is defined by

$$\tau_{jk} = P_k/V^{2/3} = \tau_{kj} = P_j/V^{2/3} \ . \tag{28}$$

Substituting Equations 10 and 23 into Equation 27, we obtain

$$\gamma_{jk} = \frac{C_k V^2 S(j,k)}{96 B D^2 L^2 I'^2} \ , \tag{29}$$

where $S(j, k)$ is given by

$$S(j,k) = \left(K_k m_{kk} + K_j m_{kj} + K_k \frac{C_j}{C_k} m_{jk} + K_j \frac{C_j}{C_k} m_{jj}\right) \Big/ K_k K_j$$
$$= \left(K_k\left(m_{kk} + \frac{C_j}{C_k} m_{jk}\right) + K_j\left(m_{kj} + \frac{C_j}{C_k} m_{jj}\right)\right) \Big/ K_k K_j \ . \tag{30}$$

If the bending modulus B of the fiber is written as

$$B = E_f I_f = E_f \eta \pi D^4/64 \ , \tag{31}$$

where E_f = Young's modulus of the fiber, I_f = moment of inertia of the fiber, and η = a shape factor for fiber bending, then the shear modulus G_{jk} can be written as

$$G_{jk} = \frac{\tau_{jk}}{\gamma_{jk}} = \frac{192 E_f \eta V_f^3 K_j}{\pi^2 S(j,k) I'^2}\left\{(1 - SNG) + SNG \frac{C_{sk}}{C_k}\right\} \ , \tag{32}$$

which should be equal to

$$G_{kj} = \frac{\tau_{kj}}{\gamma_{kj}} = \frac{192 E_f \eta V_f^3 K_k}{\pi^2 S(k,j) I'^2}\left\{(1 - SNG) + SNG \frac{C_{sj}}{C_j}\right\} \ . \tag{33}$$

Evaluation of the Theory

For the Special Case Where Fiber Slippage is Excluded

When the mechanism of fiber slippage is excluded, that is, when $SNG = 0$, it is possible to evaluate the theory using the rules derived from ordinary continuum mechanics. All relevant equations become

$$P_j = n_{bk} C_j \ , \tag{34}$$

$$P_k = n_{bj} C_k \ , \tag{35}$$

and from Equation 22,

$$C_j/C_k = K_j/K_k \ . \tag{36}$$

Thus from Equations 32 and 33, we have

$$G_{jk} = \frac{192 E_f \eta V_f^3 K_j}{\pi^2 S(j,k) I^2} \ , \tag{37}$$

and

$$G_{kj} = \frac{192 E_f \eta V_f^3 K_k}{\pi^2 S(k,j) I^2} \ . \tag{38}$$

Taking Equation 36 into Equation 30 gives

$$\frac{K_j}{S(j,k)} = \frac{m_{kk} + m_{jk} + m_{jj} + m_{kj}}{K_j K_k} = \frac{K_k}{S(k,j)} \ . \tag{39}$$

Hence

$$G_{jk} = G_{kj} \ . \tag{40}$$

Furthermore, we can test Equation 37 for some simple distribution functions.

Random Fibrous Assembly

In this case the assembly should behave as an isotropic continuum. From Komori and Makishima [7], the specific form of the density function is

$$\Omega(\theta, \phi) \sin\theta = \sin\theta/2\pi \quad . \tag{41}$$

We can also calculate that

$$k_1 = k_2 = k_3 = 1/8 \quad , \tag{42}$$

$$I = \pi/4 \quad , \tag{43}$$

$$m_{jj} = 1/6 \quad , \tag{44}$$

and

$$m_{jk} = \pi/6 \quad . \tag{45}$$

Since $K_j = K_k$, we have

$$C_j = C_k \quad , \tag{46}$$

and

$$S(j, k) = \frac{64}{3}\left(1 + \frac{1}{\pi}\right) \quad , \tag{47}$$

whence

$$G_{jk} = \frac{9}{16} E_f \eta V_f^3 \frac{1}{(1 + 1/\pi)} \quad . \tag{48}$$

Now G for an isotropic solid is related to the tensile modulus E and Poisson's ratio ν by the relationship

$$G = E/2(1 + \nu) \quad . \tag{49}$$

If we use the results of Lee and Lee [8] and multiply their result by the factor 4, as follows from the use of the revised beam equation, then we derive

$$E = \frac{9}{8} E_f \eta V_f^3 \quad , \tag{50}$$

and

$$\nu = 1/\pi \quad . \tag{51}$$

Substituting these values into Equation 49 produces Equation 48, which is the required proof.

Uniaxially Oriented Fiber Assembly

The density function proposed by Lee and Lee [8] for this transversely symmetrical case is

$$\Omega(\theta, \phi) \sin\theta = \frac{1}{2\pi \sin q} [\delta(\theta - q) + \delta\{\theta - (\pi - q)\}] \sin\theta \quad , \tag{52}$$

where $\delta(\theta)$ represents the delta function and q is the helix angle of the helically crimped oriented fibers ($0 < q < \pi/2$).

The relevant parameters may be calculated as follows:

$$\begin{aligned} k_1 &= k_2 = \sin q/2\pi \\ k_3 &= \cos q/4 \\ m_{11} &= m_{22} = (2 - \sin^2 q)/8 \\ m_{33} &= \sin^2 q/4 \\ m_{12} &= m_{21} = \sin^2 q/4\pi \\ m_{23} &= m_{32} = m_{31} = m_{13} = \sin q \cos q/2\pi \quad . \end{aligned} \tag{53}$$

The recalculated compression moduli and Poisson's ratios are as follows [8]:

$$E_{11} = E_{22} = \frac{384}{\pi^4} E_f \eta V_f^3 \frac{\sin^2 q}{2 - \sin^2 q} I^2 \quad , \tag{54}$$

$$E_{33} = \frac{48}{\pi^2} E_f \eta V_f^3 \cot^2 q I^2 \quad , \tag{55}$$

$$\nu_{12} = \nu_{21} = \frac{2 \sin^2 q}{\pi(2 - \sin^2 q)} \quad , \tag{56}$$

and

$$\nu_{13} = \nu_{23} = \frac{8 \sin^2 q}{\pi^2(2 - \sin^2 q)} \quad . \tag{57}$$

By using the results in Equation 53, we can calculate the values of $S(j, k)$ in terms of different directions:

$$S(1, 2) = S(2, 1) = \frac{2\pi^2 - \pi^2 \sin^2 q + 2\pi \sin^2 q}{\sin^2 q} \quad , \tag{58}$$

and

$$S(1, 3) = S(2, 3) = S(3, 1) = S(3, 2) = \frac{2\pi + \pi \sin^2 q + 8 \sin q \cos q}{\sin q \cos q} \quad . \tag{59}$$

So the shear moduli can accordingly be written as

$$G_{12} = G_{21} = \frac{384}{\pi^4} E_f \eta V_f^3 I^2 \frac{\sin^2 q}{2(2 - \sin^2 q) + \frac{4}{\pi} \sin^2 q} \quad , \tag{60}$$

and

$$G_{13} = G_{31} = G_{23} = G_{32}$$
$$= \frac{192}{\pi^2} E_f \eta V_f^3 I^2 \frac{\sin q \cos q}{2\pi + \pi \sin^2 q + 8 \sin q \cos q} \quad . \tag{61}$$

There is a similar relationship between the shear and compression moduli and Poisson's ratio for the transversely symmetric continuum:

$$G_{12} = E_{11}/2(1 + \nu_{12}) \quad . \tag{62}$$

By inserting Equations 54 and 56 into Equation 62, we reproduce Equation 60, *i.e.*, the equation we have derived.

FOR THE CASE WHERE FIBER SLIPPAGE OCCURS

In the case where fiber slippage does occur, symmetry relations between shear and compression moduli and Poisson's ratio are not necessarily valid. However, in any case,

$$G_{jk} \text{ should equal } G_{kj} \quad , \tag{63}$$

and we will prove this below.

According to the definition of the shear modulus, Equation 63 can be written as

$$G_{jk} = \frac{\tau_{jk}}{\gamma_{jk}} = \frac{P_k}{V^{2/3}\gamma_{jk}}$$
$$= \frac{P_j}{V^{2/3}\gamma_{kj}} = G_{kj} \quad . \tag{64}$$

Taking γ_{jk} in Equation 29 into Equation 64 and eliminating identical terms gives

$$\frac{P_k}{C_k S(j, k)} = \frac{P_j}{C_j S(k, j)} \quad , \tag{65}$$

where

$$S(j, k)$$
$$= \frac{K_k m_{kk} + K_j m_{kj} + K_k \frac{C_j}{C_k} m_{jk} + K_j \frac{C_j}{C_k} m_{jj}}{K_k K_j} \quad , \tag{66}$$

and

$$S(k, j)$$
$$= \frac{K_j m_{jj} + K_k m_{jk} + K_j \frac{C_k}{C_j} m_{kj} + K_k \frac{C_k}{C_j} m_{kk}}{K_k K_j} \quad . \tag{67}$$

For shear equilibrium, the shear forces P_k and P_j should be equal, and Equation 65 can therefore be written as

$$C_k S(j, k) = K_k C_k m_{kk} + K_j C_k m_{kj}$$
$$+ K_k C_j m_{jk} + K_j C_j m_{jj}$$
$$= C_j S(k, j) \quad . \tag{68}$$

Hence the condition of Equation 63 is satisfied.

Discussion

This paper reveals that the low-strain response of fibrous assemblies to shear can be modeled using a combined mechanism of fiber bending and slippage at the contact points. As such, the internal deformation is similar to that created by external compression. The derivation is complex, but the results can be checked using simple symmetry rules relating the mechanical constants for special cases. The accuracy of the derived modulus value may be inferred from the earlier compression experiments of Lee and Lee with which it is related via these symmetry requirements; however, more precise evaluation of the theory must await the completion of the full tangent compliance matrix for the unit fibrous cell. Once all the mechanical interactions are modeled for the three-dimensional load case, it will be possible to apply the full theory directly to practical loading cases and more importantly to yarns and higher order textile structures.

ACKNOWLEDGMENTS

Dr. N. Pan gratefully acknowledges the financial assistance of the New Zealand Wool Board, which has enabled him to spend a period of secondment to WRONZ.

Literature Cited

1. Backer, S., The Mechanics of Bent Yarns, *Textile Res. J.* **22,** 668–681 (1952).
2. Carnaby, G. A., and Curiskis, J. I., The Tangent Compliance of Staple-fibre Bundles in Tension, *J. Textile Inst.* **78,** 293–305 (1987).
3. Carnaby, G. A., and Pan, N., Theory of the Compression Hysteresis of Fibrous Assemblies, *Textile Res. J.* **59,** 275–284 (1989).
4. Curiskis, J. I., and Carnaby, G. A., Continuum Mechanics of the Fiber Bundle, *Textile Res. J.* **55,** 334–344 (1985).
5. Grosberg, P., The Mechanical Properties of Woven Fabrics, Part II: The Bending of Woven Fabrics, *Textile Res. J.* **36,** 205–211 (1966).
6. Grosberg, P., and Smith, P. A., The Strength of Slivers of Relatively Low Twist, *J. Textile Inst.* **57,** T15–T23 (1966).

7. Komori, T., and Makishima, K., Numbers of Fiber-to-fiber Contacts in General Fiber Assemblies, *Textile Res. J.* **47**, 13–17 (1977).
8. Lee, D. H., and Lee, J. K., Initial Compressional Behaviour of Fibre Assembly, in: "Objective Measurement: Applications to Product Design and Process Control," S. Kawabata, R. Postle, and M. Niwa, Eds., Textile Machinery Society of Japan, Osaka, 1985, pp. 613–622.
9. Pan, N., and Carnaby, G. A., The Initial Shear Modulus of a Unit Cell of Wool Fibres, WRONZ Communication No. C106, 1988.
10. Valliappan, S., "Continuum Mechanics Fundamentals," A. A. Balkema, Rotterdam, 1981, p. 51.
11. van Wyk, C. M., Note on the Compressibility of Wool, *J. Textile Inst.* **37**, T285–T292 (1946).

Manuscript received May 13, 1988; accepted July 15, 1988.